Complex analysis, homework 1 plus 2, solutions.

Exercise 1.[12 points] Compute the following quantities. Show your steps.

(1) $(3-i)(-2+5i)-3+2i$
(2) $\dfrac{-3+2i}{2-i}$
(3) $(1+i)^3$

Solution.

(1) $(3-i)(-2+5i)-3+2i = (-6+2i+15i+5)-3+2i = -4+19i$.
(2) $\dfrac{-3+2i}{2-i} = \dfrac{-3+2i}{2-i}\cdot\dfrac{2+i}{2+i} = \dfrac{-6-3i+4i-2}{2^2-i^2} = \dfrac{-8+i}{4+1} = -\dfrac{8}{5}+\dfrac{1}{5}i$.
(3) We can use the usual formula for the $(a+b)^3$: we have

$$(1+i)^3 = 1^3+3\cdot 1^2\cdot i+3\cdot 1\cdot i^2+i^3 = 1+3i-3-i = -2+2i.$$

Exercise 2. [4 points] Which of the points $z_1 = 3+6i$ and $z_2 = 5-4i$ is closer to the origin?

Solution. On the one hand, $|z_1| = \sqrt{3^2+6^2} = \sqrt{9+36} = \sqrt{45}$. On the other hand, $|z_2| = \sqrt{5^2+(-4)^2} = \sqrt{25+16} = \sqrt{41}$. Therefore, $|z_2| < |z_1|$ so z_2 is closer to the origin than z_1.

Exercise 3. [6 points]

(1) Show that, for any $z\in\mathbb{C}$, $z^2+1 = (z-i)(z+i)$.
(2) Prove that the equation $z^2+1=0$ has exactly two solutions, which are i and $-i$.

Solution.

(1) Let $z\in\mathbb{C}$. Then, using the formula $(a-b)(a+b) = a^2-b^2$, we have $(z-i)(z+i) = z^2-i^2 = z^2+1$.
(2) Let $z\in\mathbb{C}$. Then

$$z^2+1=0 \quad\Leftrightarrow\quad (z-i)(z+i)=0 \quad\Leftrightarrow\quad \begin{cases} z-i=0 \\ \text{or} \\ z+i=0 \end{cases} \quad\Leftrightarrow\quad \begin{cases} z=i \\ \text{or} \\ z=-i. \end{cases}$$

Therefore, the equation $z^2+1=0$ has exactly two solutions, which are i and $-i$.

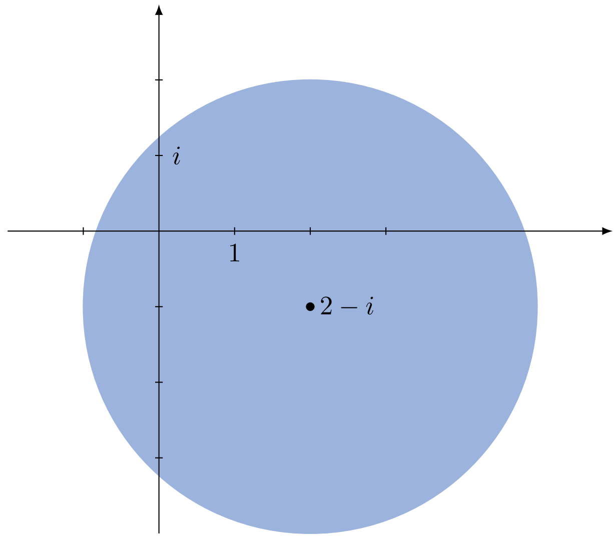
We used here in the second equivalence the following fact:

Fact: Let $z_1, z_2\in\mathbb{C}$. Then, $z_1z_2 = 0$ if and only if $z_1 = 0$ or $z_2 = 0$.

Note that we have already proved in class that if $z_1z_2 = 0$ then $z_1 = 0$ or $z_2 = 0$. The other direction of the statement is obvious.

Exercise 4. [4 points] Sketch the region in the complex plane $\{z\in\mathbb{C} : |z-2+i|\le 3\}$, that is the set of all points z such that $|z-2+i|\le 3$.

Solution. Note that $|z-2+i| = |z-(2-i)|$ is the distance between z and $2-i$. So the region $\{z\in\mathbb{C} : |z-2+i|\le 3\}$ is the closed (that is including the boundary) disk centered at $2-i$ with radius 3. The region is pictured in blue below.



Exercise 5. [4 points] Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be complex numbers. Express $\mathrm{Re}(z_1\overline{z_2})$ in terms of x_1, x_2, y_1, y_2. What does it represent for the vectors z_1 and z_2?

Solution. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be complex numbers. We have

$$z_1\overline{z_2} = (x_1 + iy_1)(x_2 - iy_2) = x_1x_2 + iy_1x_2 - x_1iy_2 - i^2y_1y_2 = (x_1x_2 + y_1y_2) + i(y_1x_2 - x_1y_2)$$

and therefore

$$\mathrm{Re}(z_1\overline{z_2}) = x_1x_2 + y_1y_2.$$

This is the scalar product (or dot product) between the vectors z_1 and z_2.

Exercise 6. [4 points] Let $z_1, z_2 \in \mathbb{C}$ be in the upper left quarter plane (that is with negative real part and positive imaginary part). Prove that

$$\mathrm{Arg}(z_1z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) - 2\pi.$$

Solution. Let $z_1, z_2 \in \mathbb{C}$ be in the upper left quarter plane. Note that this implies that $\mathrm{Arg}(z_1) \in (\frac{\pi}{2}, \pi)$ and $\mathrm{Arg}(z_2) \in (\frac{\pi}{2}, \pi)$. Moreover,

$$z_1z_2 = (|z_1|e^{i\,\mathrm{Arg}(z_1)})(|z_2|e^{i\,\mathrm{Arg}(z_2)}) = (|z_1| \cdot |z_2|)e^{i(\mathrm{Arg}(z_1)+\mathrm{Arg}(z_2))} = |z_1z_2|e^{i(\mathrm{Arg}(z_1)+\mathrm{Arg}(z_2)-2\pi)},$$

using $e^{-i2\pi} = 1$ in the last equality. Therefore, $\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) - 2\pi$ is an argument of z_1z_2. But $\mathrm{Arg}(z_1)+\mathrm{Arg}(z_2) \in (\pi, 2\pi)$, so $\mathrm{Arg}(z_1)+\mathrm{Arg}(z_2)-2\pi \in (-\pi, 0)$ so in particular it is in $(-\pi, \pi]$. Hence $\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) - 2\pi$ is the principal argument of z_1z_2.

Exercise 7. [4 points] Let $w, z \in \mathbb{C}$ with $|w| = 1$ and $z \neq w$. Prove that

$$\left|\frac{w - z}{1 - \overline{w}z}\right| = 1.$$

Solution. Note that $w\overline{w} = |w|^2 = 1$. Therefore,

$$\begin{aligned}
|1 - \overline{w}z| &= |1 - \overline{w}z| \cdot |w| && \text{(using that } |w| = 1\text{)}\\
&= |(1 - \overline{w}z)w| && \text{(using that } |z_1z_2| = |z_1||z_2|\text{)}\\
&= |w - w\overline{w}z| \\
&= |w - z| && \text{(using that } w\overline{w} = 1\text{)}
\end{aligned}$$

This implies in particular that $1-\overline{w}z \neq 0$, because $|w-z| \neq 0$ (since $w \neq z$). Therefore, the left-hand side in the equation we want to prove makes sense! Moreover, we have

$$\left|\frac{w-z}{1-\overline{w}z}\right| = \frac{|w-z|}{|1-\overline{w}z|} = \frac{|w-z|}{|w-z|} = 1,$$

where we used the previous calculation in the second inequality.

Exercise 9. [4 points] Prove that for any z with modulus $R > 1$, one has

$$\left|\frac{z^4+\mathrm{i}z}{z^2+z+1}\right| \leq \frac{R^4+R}{(R-1)^2}.$$

Solution. We have

$$\left|\frac{z^4+\mathrm{i}z}{z^2+z+1}\right| = \left|\frac{(z^4+\mathrm{i}z)(z-1)}{(z^2+z+1)(z-1)}\right| = \left|\frac{(z^4+\mathrm{i}z)(z-1)}{z^3-1}\right| \leq \frac{(R^4+R)(R+1)}{R^3-1} = \frac{(R^4+R)(R+1)}{(R^2+R+1)(R-1)} \leq \frac{R^4+R}{(R-1)^2},$$

where the last inequality is due to $R^2-1 \leq R^2+R+1$, obviouly correct.