Complex analysis, homework 3 due February 8th.

**Exercise 1.** [4 points] Calculate \((-2 + 2i)^{10}\). Give your result in the form \(x + iy\) with \(x\) and \(y\) real numbers. Show your steps.
*Remark:* We have seen a method in class for this, do not expand directly \((-2+2i)^{10}\).

**Exercise 2.** [6 points]
1. Find the fourth roots of \(i\). Give them in exponential forms and then represent them on a picture. Highlight the principal fourth root.
2. Find the third roots of \(-8 + 8\sqrt{3}i\). Give them in exponential forms and then represent them on a picture. Highlight the principal third root.

**Exercise 3.** [4 points] We consider the following transformation \(z \rightarrow 2e^{i\pi/4}(z-1+i)\).
Describe its effect on a point \(z\) of the complex plane in words (there should be three successive simple steps). Illustrate it with a picture in the case \(z = 2 + i\) (that is represent \(z\) and \(2e^{i\pi/4}(z-1+i)\), as well as the results of the successive steps described earlier).

**Exercise 4.** [4 points] Prove that \(\lim_{z \to 1-2i} \frac{2z + 1}{iz + 1}\) exists and give its value in the form \(x + iy\).

**Exercise 5.** [5 points] Let \(f\) be a function defined on \(\mathbb{C}\). We say that \(f\) is Lipschitz on \(\mathbb{C}\) if there exists \(K > 0\) such that, for any \(z, z' \in \mathbb{C}\),
\[|f(z) - f(z')| \leq K|z - z'|.\]
Prove that, if \(f\) is Lipschitz on \(\mathbb{C}\), then \(f\) has a limit at any point in \(\mathbb{C}\).

**Exercise 6.** [5 points] Prove that \(\lim_{z \to -1} \operatorname{Arg}(z)\) does not exist.

**Exercise 7.** [8 points] Let \(z_0 \in \mathbb{C}\). **Prove or disprove** the following statements:
1. Let \(f\) and \(g\) be functions defined on a deleted neighborhood of \(z_0\).
   If \(\lim_{z \to z_0} f(z) = \infty\) and \(\lim_{z \to z_0} g(z) = \infty\), then \(\lim_{z \to z_0} (f(z) + g(z)) = \infty\).
2. Let \(f\) and \(g\) be functions defined on a deleted neighborhood of \(z_0\).
   If \(\lim_{z \to z_0} f(z) = \infty\) and \(\lim_{z \to z_0} g(z) = \infty\), then \(\lim_{z \to z_0} (f(z) \times g(z)) = \infty\).
*Remark:* In order to disprove a result, you have to give a counterexample.