Complex analysis, homework 4 due February 15th.

Exercise 1.[8 points] For the following functions, say at which points they are differentiable and find their derivatives. Show your steps.

(1) \( f(z) = \frac{z^2}{iz + 1} \)

(2) \( f(z) = z(z^2 + iz)^5 \)

Exercise 2.[5 points] Let \( z_0 \in \mathbb{C} \). Let \( f \) be a function differentiable at \( z_0 \). For any \( z \in \mathbb{C} \) such that \( f(z) \) is defined, we set \( g(z) = f(z) \).

Prove that \( g \) is differentiable at \( z_0 \) and express \( g'(z_0) \) in terms of \( f'(z_0) \).

Exercise 3.[8 points] Let \( f(z) = z \text{Im}(z) \) for \( z \in \mathbb{C} \). Find the points \( z \in \mathbb{C} \) where \( f \) is differentiable and find its derivative \( f'(z) \) at these points. For all the other points in the complex plane, prove that \( f \) is not differentiable at these points.

Exercise 4.[9 points] Let \( f \) be a function differentiable on \( \mathbb{C} \).

(1) Prove that if \( \text{Re}(f) \) is constant on \( \mathbb{C} \), then \( f \) is constant on \( \mathbb{C} \).

(2) Prove that if \( |f| \) is constant on \( \mathbb{C} \), then \( f \) is constant on \( \mathbb{C} \).

Hint: Use the Cauchy-Riemann equations. You can use the following fact: if a real-valued function on \( \mathbb{R}^2 \) has its both partial derivatives that are zero on \( \mathbb{R}^2 \), then this function is constant on \( \mathbb{R}^2 \). For (b), you can start by squaring the modulus and differentiate either with respect to \( x \) or with respect to \( y \).