Complex analysis, homework 4, solutions.

Exercise 1. For the following functions, say at which points they are differentiable and find their derivatives. Show your steps.

(1) $f(z) = \dfrac{z^2}{iz+1}$
(2) $f(z) = z(z^2+iz)^5$

Solution.

(1) First note that $f(z)$ is defined iff $iz+1 \neq 0$, which is equivalent to $z \neq i$. For $z \neq i$, the numerator and the denominator are differentiable at z as polynomials and the denominator is non-zero, so by the quotient rule, f is differentiable at z and

$$f'(z) = \frac{2z(iz+1) - iz^2}{(iz+1)^2} = \frac{iz^2+2z}{(iz+1)^2}.$$

(2) The function f is a polynomial so it is differentiable everywhere. TUsing the product rule and then the chain rule, we get

$$\begin{aligned} f'(z) &= (z^2+iz)^5 + z \cdot \frac{\mathrm{d}}{\mathrm{d}\,z}(z^2+iz)^5 \\ &= (z^2+iz)^5 + z \cdot (2z+i) \cdot 5(z^2+iz)^4 \\ &= (z^2+iz)^4(z^2+iz+5z(2z+i)) \\ &= (z^2+iz)^4(11z+6i)z. \end{aligned}$$

Exercise 2[5 points] Let $z_0 \in \mathbb{C}$. Let f be a function differentiable at z_0. For any $z \in \mathbb{C}$ such that $f(\overline{z})$ is defined, we set

$$g(z) = \overline{f(\overline{z})}.$$

Prove that g is differentiable at $\overline{z_0}$ and express $g'(\overline{z_0})$ in terms of $f'(z_0)$.

Solution. For $h \in \mathbb{C}$ in a small enough neighborhood of 0, we have

$$\begin{aligned} \frac{g(\overline{z_0}+h) - g(\overline{z_0})}{h} &= \frac{\overline{f\left(\overline{\overline{z_0}+h}\right)} - \overline{f\left(\overline{\overline{z_0}}\right)}}{h} && \text{(by definition of } g) \\ &= \frac{\overline{f\left(z_0+\overline{h}\right) - f(z_0)}}{\overline{\overline{h}}} && \text{(using } \overline{\overline{z}} = z \text{ and } \overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}) \\ &= \overline{\left(\frac{f\left(z_0+\overline{h}\right) - f(z_0)}{\overline{h}}\right)} && \text{(using } \overline{z_1/z_2} = \overline{z_1}/\overline{z_2}). \end{aligned}$$

We know that $\lim_{h\to 0} \overline{h} = 0$ and $\lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h} = f'(z_0)$, so by composition of limits $\lim_{h\to 0} \frac{f(z_0+\overline{h})-f(z_0)}{\overline{h}} = f'(z_0)$. Moreover, using that the limit of the conjugate equals the conjugate of the limit, we get

$$\lim_{h\to 0} \frac{g(\overline{z_0}+h) - g(\overline{z_0})}{h} = \overline{f'(z_0)}.$$

So g is differentiable at $\overline{z_0}$ and $g'(\overline{z_0}) = \overline{f'(z_0)}$.

Exercise 3.[8 points] Let $f(z) = z\,\mathrm{Im}(z)$ for $z \in \mathbb{C}$. Find the points $z \in \mathbb{C}$ where f is differentiable and find its derivative $f'(z)$ at these points. For all the other points in the complex plane, prove that

f is not differentiable at these points.

Solution. For any $z = x + iy \in \mathbb{C}$, we can write $f(z) = (x + iy)y = xy + iy^2 = u(x, y) + iv(x, y)$, with $u(x, y) = xy$ and $v(x, y) = y^2$. The functions u and v have partial derivatives everywhere, which are

$$u_x(x, y) = y \qquad u_y(x, y) = x \qquad v_x(x, y) = 0 \qquad v_y(x, y) = 2y.$$

Hence, for $(x, y) \in \mathbb{R}^2$, we have

$$\begin{cases} u_x(x, y) = v_y(x, y) \\ u_y(x, y) = -v_x(x, y) \end{cases} \quad \Leftrightarrow \quad \begin{cases} y = 2y \\ x = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} y = 0 \\ x = 0 \end{cases}$$

Hence, Cauchy-Riemann equations are only satisfied at $(0, 0)$. So for any $z \neq 0$, f is not differentiable at z.

We now want to prove f is differentiable at 0.

Approach 1 (most efficient): Since u and v have partial derivatives in a neighborhood of $(0, 0)$, they are all continuous at $(0, 0)$ and Cauchy-Riemann equations are satisfied at $(0, 0)$, by the theorem of Section 23, we get that f is differentiable at 0 and

$$f'(0) = u_x(0, 0) + iv_x(0, 0) = 0.$$

Approach 2 (using only results of past week): We use the definition of differentiability. For $h \in \mathbb{C}$,

$$\frac{f(0 + h) - f(0)}{h} = \frac{h \operatorname{Im}(h) - 0}{h} = \operatorname{Im}(h).$$

Hence

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = 0.$$

So f is differentiable at 0 and $f'(0) = 0$.

Exercise 4[9 points] Let f be a function differentiable on $\mathbb{C}$.

(1) Prove that if $\operatorname{Re}(f)$ is constant on $\mathbb{C}$, then f is constant on $\mathbb{C}$.
(2) Prove that if $|f|$ is constant on $\mathbb{C}$, then f is constant on $\mathbb{C}$.

Hint: Use the Cauchy-Riemann equations. You can use the following fact: if a real-valued function on $\mathbb{R}^2$ has its both partial derivatives that are zero on $\mathbb{R}^2$, then this function is constant on $\mathbb{R}^2$. For (b), you can start by squaring the modulus and differentiate either with respect to x or with respect to y.

Solution For both parts, we write $f(z) = u(x, y) + iv(x, y)$ for any $z = x + iy \in \mathbb{C}$. Since f is differentiable on $\mathbb{C}$, u and v have partial derivatives on $\mathbb{R}^2$ and the Cauchy-Riemann equations are true: $u_x = v_y$ and $u_y = -v_x$.

(1) We assume $\operatorname{Re}(f)$ is constant on $\mathbb{C}$, that is u is constant on $\mathbb{R}^2$. Since u is constant, $u_x = 0$ and $u_y = 0$. It follows from the Cauchy-Riemann equations that $v_x = 0$ and $v_y = 0$. By the fact in the hint, we deduce that v is constant on $\mathbb{R}^2$. Therefore f is constant on $\mathbb{C}$.
(2) We assume $|f|$ is constant on $\mathbb{C}$. Hence $|f|^2 = u^2 + v^2$ is also constant on $\mathbb{C}$. So differentiating with respect to x and with respect to y, we get

$$2u_x u + 2v_x v = 0 \qquad \text{and} \qquad 2u_y u + 2v_y v = 0.$$

Using the Cauchy-Riemann equations in the second equation, we get

$$\begin{cases} u_x u + v_x v = 0 \\ -v_x u + u_x v = 0. \end{cases} \tag{0.1}$$

Multiplying the first equation by u and the second one by v and then summing them, we get

$$(u^2 + v^2)u_x = 0. \tag{0.2}$$

If for some $(x, y) \in \mathbb{R}^2$, $u^2(x, y) + v^2(x, y) = 0$, then this means that $|f(x + iy)| = 0$, but $|f|$ is constant on $\mathbb{C}$ so for any $z' \in \mathbb{C}$, we have $|f(z')| = 0$ and therefore $f(z') = 0$. In particular f is constant on $\mathbb{C}$.

Otherwise u^2+v^2 does not vanish on $\mathbb{R}^2$ so we can deduce from (0.2) that $u_x = 0$ everywhere. But, back to (0.1), multiplying the first equation by v and the second one by u and then subtracting them, we get $(u^2 + v^2)v_x = 0$ which implies that $v_x = 0$ everywhere. And using the Cauchy-Riemann equations, we also have $u_y = 0$ and $v_y = 0$. By the fact on the hint, we deduce that u and v are constant on $\mathbb{R}^2$. Therefore f is constant on $\mathbb{C}$.