Complex analysis, homework 5, solutions

Exercise 1[5 points] Prove the function defined by $f(z) = x^2 - y^2 + y + 2 + ix(2y-1)$ for $z = x + iy$ is entire and find $f'(z)$.

Solution. We write $f(z) = u(x, y) + iv(x, y)$ with $u(x, y) = x^2 - y^2 + y + 2$ and $v(x, y) = 2xy - x$. Note that u and v are two-variable polynomials so they have partial derivatives everywhere and these partial derivatives are continuous everywhere. Moreover,

$$u_x(x, y) = 2x = v_y(x, y)$$
$$u_y(x, y) = -2y + 1 = -(2y - 1) = -v_x(x, y).$$

Therefore, the Cauchy-Riemann Equations are satisfied everywhere. We can apply the theorem in Section 23 to conclude that f is differentiable on $\mathbb{C}$ and therefore entire. Moreover,

$$f'(z) = u_x(x, y) + iv_x(x, y) = 2x + i(2y - 1) = 2z - i.$$

Exetcise 2.[5 points] Compute the following quantities (that is express them in $x + iy$ form):

(1) $\exp(2 + i\frac{5\pi}{6})$;
(2) $\log((-e + ei)/\sqrt{2})$ and $\operatorname{Log}((-e + ei)/\sqrt{2})$.

Solution.

(1) $\exp(2 + i\frac{5\pi}{6}) = e^2 \cdot e^{i\frac{5\pi}{6}} = e^2(\cos(\frac{5\pi}{6}) + i\sin(\frac{5\pi}{6})) = e^2(-\frac{\sqrt{3}}{2} + i\frac{1}{2}) = -\frac{\sqrt{3}}{2}e^2 + i\frac{e^2}{2}$.
(2) Let $z = (-e + ei)/\sqrt{2}$. Then

$$|z| = \sqrt{\left(\frac{-e}{\sqrt{2}}\right)^2 + \left(\frac{e}{\sqrt{2}}\right)^2} = \sqrt{\frac{e^2}{2} + \frac{e^2}{2}} = \sqrt{e^2} = e.$$

Therefore, $z = |z|(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = |z|(\cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4})) = e \cdot e^{i\frac{3\pi}{4}}$. We get

$$\log(z) = \ln|z| + i\arg(z) = 1 + i\left(\frac{3\pi}{4} + 2k\pi\right), \quad k \in \mathbb{Z}.$$

Since $\frac{3\pi}{4} \in (-\pi, \pi]$, we have $\operatorname{Arg}(z) = \frac{3\pi}{4}$ and therefore

$$\operatorname{Log}(z) = 1 + i\frac{3\pi}{4}.$$

Execise 3.[3 points] Let $z \in \mathbb{C}$. Prove that $\overline{\exp(z)} = \exp(\overline{z})$.

Solution. Write $z = x + iy$, with $x, y \in \mathbb{R}$. Then

$$\overline{\exp(z)} = \overline{e^x \cdot e^{iy}} = \overline{e^x} \cdot \overline{e^{iy}}.$$

Since e^x is real, we have $\overline{e^x} = e^x$. On the other hand, since $e^{iy} = \cos y + i \sin y$,

$$\overline{e^{iy}} = \cos y - i \sin y = \cos(-y) + i \sin(-y) = e^{-iy} \quad \text{(This is a useful formula!)}$$

so we get

$$\overline{\exp(z)} = e^x \cdot e^{-iy} = e^{x-iy} = \exp(\overline{z}).$$

Exercise 4.[4 points] Solve the equation $e^{2z} + 1 = i$.

Solution. Let $z = x + iy \in \mathbb{C}$. We have

$$\begin{aligned}
e^{2z} + 1 = i \quad &\Leftrightarrow \quad e^{2z} = -1 + i \\
&\Leftrightarrow \quad e^{2z} = \sqrt{2} e^{i\frac{3\pi}{4}} \\
&\Leftrightarrow \quad e^{2x+i2y} = e^{\ln\sqrt{2} + i\frac{3\pi}{4}} \\
&\Leftrightarrow \quad \begin{cases} 2x = \ln\sqrt{2} \\ 2y = \frac{3\pi}{4} + 2k\pi, \text{ for some } k \in \mathbb{Z} \end{cases} \\
&\Leftrightarrow \quad \begin{cases} x = \frac{1}{4}\ln 2 \\ y = \frac{3\pi}{8} + k\pi, \text{ for some } k \in \mathbb{Z} \end{cases} \\
&\Leftrightarrow \quad z = \frac{1}{4}\ln 2 + i\left(\frac{3\pi}{8} + k\pi\right), \text{ for some } k \in \mathbb{Z}.
\end{aligned}$$

So the set of solutions to the equation is $\{\frac{1}{4}\ln 2 + i(\frac{3\pi}{8} + k\pi) : k \in \mathbb{Z}\}$.

Exercise 5.[6 points] Prove that

(1) $\operatorname{Log}((1-i)^2) = 2\operatorname{Log}(1-i)$;
(2) $\operatorname{Log}((1+i\sqrt{3})^4) \neq 4\operatorname{Log}(1+i\sqrt{3})$.

Solution.

(1) First note that $1 - i = \sqrt{2}e^{-i\frac{\pi}{4}}$, where $-\frac{\pi}{4}$ is its principal argument. So we have

$$\operatorname{Log}(1-i) = \ln\sqrt{2} - i\frac{\pi}{4} = \frac{1}{2}\left(\ln 2 - i\frac{\pi}{2}\right).$$

Moreover, $(1-i)^2 = (\sqrt{2}e^{-i\frac{\pi}{4}})^2 = 2e^{-i\frac{\pi}{2}}$, where $-\frac{\pi}{2}$ is its principal argument. So we have

$$\operatorname{Log}((1-i)^2) = \ln 2 - i\frac{\pi}{2}.$$

This proves that $\operatorname{Log}((1-i)^2) = 2\operatorname{Log}(1-i)$.

(2) First note that $1 + i\sqrt{3} = 2(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 2e^{i\frac{\pi}{3}}$, where $\frac{\pi}{3}$ is its principal argument. Therefore,

$$\operatorname{Log}(1+i\sqrt{3}) = \ln 2 + i\frac{\pi}{3}$$

Moreover, $(1+i\sqrt{3})^4 = 2^4 e^{i\frac{4\pi}{3}} = 2^4 e^{-i\frac{2\pi}{3}}$, where $-\frac{2\pi}{3}$ is its principal argument. So we get

$$\operatorname{Log}((1+i\sqrt{3})^4) = \ln(2^4) - i\frac{2\pi}{3} = 4\ln 2 - i\frac{2\pi}{3} \neq 4\ln 2 + i\frac{4\pi}{3} = 4\operatorname{Log}(1+i\sqrt{3}).$$

Exercise 6.[7 points] Recall that for any $z \neq 0$, we define $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z)$. Let $D = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$.

(1) Using a geometric argument, express $\operatorname{Arg}(z)$ for $z = x + iy \in D$ in terms of $\cos^{-1}$, x and y. Explain why this formula does not work for all $z \neq 0$.
(2) Using the theorem of Section 23, prove that Log is analytic on D and that $\operatorname{Log}'(z) = 1/z$ for any $z \in D$.

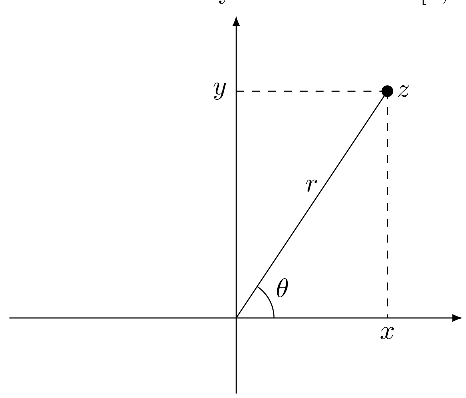
Reminder: $\dfrac{\mathrm{d}}{\mathrm{d}t}\cos^{-1}(t) = -\dfrac{1}{\sqrt{1-t^2}}$.

Solution.

(1) Let $z \neq 0$. Write $z = x + iy = re^{i\theta}$, with $r > 0$ and $\theta \in (-\pi, \pi]$ (so that $\theta = \operatorname{Arg}(z)$). Then $x = r\cos\theta$ and therefore $\cos\theta = \frac{x}{r}$ (this is always true).
Now assume that $z \in D$. In that case $\theta \in (0, \pi)$, so we have (because $\cos\colon (0,\pi) \to (-1,1)$ is bijective with inverse function arccos)

$$\cos\theta = \frac{x}{r} \quad \Leftrightarrow \quad \theta = \arccos\frac{x}{r} \quad \Leftrightarrow \quad \operatorname{Arg}(z) = \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right).$$

This formula is not true if z is in the lower half plane, because then $\theta \in (-\pi, 0)$, but the function arccos only takes values in $[0, \pi]$.



(2) We write $\operatorname{Log}(z) = u(x,y) + iv(x,y)$ with

$$u(x,y) = \ln(\sqrt{x^2+y^2}) = \frac{1}{2}\ln(x^2+y^2)$$
$$v(x,y) = \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)$$

Note that u and v have partial derivatives everywhere in D: for u, note that x^2+y^2 is always positive and ln is differentiable on $(0,\infty)$, and for v note that, since x^2+y^2 is always positive, $x/\sqrt{x^2+y^2}$ has partial derivatives and $x/\sqrt{x^2+y^2}$ takes values only in $(0,1)$, where arccos is differentiable. Since D is open, for any point in D, the partial derivatives exist in a neighborhood of this point (because there is a neighborhood of this point included in D).

The partial derivatives of u are

$$u_x(x,y) = \frac{1}{2}\frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2}$$
$$u_y(x,y) = \frac{y}{x^2+y^2}.$$

The partial derivatives of v are

$$v_x(x,y) = \arccos'\left(\frac{x}{\sqrt{x^2+y^2}}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = \frac{-1}{\sqrt{1-\frac{x^2}{x^2+y^2}}} \cdot \frac{\sqrt{x^2+y^2} - x\frac{2x}{2\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$= \frac{-1}{\sqrt{\frac{y^2}{x^2+y^2}}} \cdot \frac{(x^2+y^2)-x^2}{(x^2+y^2)^{3/2}} = \frac{-\sqrt{x^2+y^2}}{y} \cdot \frac{y^2}{(x^2+y^2)^{3/2}} = \frac{-y}{x^2+y^2}$$

$$v_y(x,y) = \arccos'\left(\frac{x}{\sqrt{x^2+y^2}}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = \frac{-1}{\sqrt{1-\frac{x^2}{x^2+y^2}}} \cdot \frac{-x\frac{2y}{2\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$= \frac{1}{\sqrt{\frac{y^2}{x^2+y^2}}} \cdot \frac{xy}{(x^2+y^2)^{3/2}} = \frac{x}{x^2+y^2}.$$

Therefore, note that the Cauchy-Riemann Equations are satisfied at any point in D. Finally note that these partial derivatives are continuous on D, because x^2+y^2 is never 0. So, we can apply the theorem in Section 23 to conclude that Log is differentiable on D. Moreover,

$$\text{Log}'(z) = u_x(x,y) + iv_x(x,y) = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = \frac{x-iy}{|z|^2} = \frac{\overline{z}}{z\overline{z}} = \frac{1}{z}.$$