Complex analysis, homework 6, solutions

Exercise 1.[7 points] Calculate the following quantities:

(1) $\sin\left(\frac{\pi}{3}+i\ln(2)\right)$;

(2) $(\sqrt{3}-i)^{2-i}$ and P.V.$(\sqrt{3}-i)^{2-i}$.

Solution.

(1) By the definition of $\sin(z)$, we have

$$\begin{aligned}\sin\left(\frac{\pi}{3}+i\ln(2)\right) &= \frac{e^{i(\frac{\pi}{3}+i\ln(2))}-e^{-i(\frac{\pi}{3}+i\ln(2))}}{2i} = \frac{1}{2i}\left(e^{-\ln(2)}e^{i\frac{\pi}{3}}-e^{\ln(2)}e^{-i\frac{\pi}{3}}\right)\\ &= \frac{1}{2i}\left(\frac{1}{2}\frac{1+i\sqrt{3}}{2}-2\frac{1-i\sqrt{3}}{2}\right) = \frac{1}{8i}\left(1+i\sqrt{3}-4(1-i\sqrt{3})\right)\\ &= \frac{1}{8i}\left(-3+5i\sqrt{3}\right) = \frac{5\sqrt{3}}{8}+\frac{3i}{8}.\end{aligned}$$

(2) First note that $\sqrt{3}-i = 2e^{-i\frac{\pi}{6}}$, where $-\frac{\pi}{6}$ is its principal argument. Therefore,

$$\begin{aligned}\log(\sqrt{3}-i) &= \ln(2)+i\left(-\frac{\pi}{6}+2k\pi\right), \quad k\in\mathbb{Z},\\ \mathrm{Log}(\sqrt{3}-i) &= \ln(2)-i\frac{\pi}{6}.\end{aligned}$$

Hence, we get

$$\begin{aligned}(\sqrt{3}-i)^{2-i} &= \exp((2-i)\log(\sqrt{3}-i))\\ &= \exp\left((2-i)\left(\ln(2)-i\frac{\pi}{6}+2ik\pi\right)\right), \quad k\in\mathbb{Z},\\ &= \exp\left(2\ln(2)-\frac{\pi}{6}-2k\pi-i\left(\ln(2)+\frac{\pi}{3}\right)+4ik\pi\right), \quad k\in\mathbb{Z},\\ &= 4\exp\left(-\frac{\pi}{6}-2k\pi-i\left(\ln(2)+\frac{\pi}{3}\right)\right), \quad k\in\mathbb{Z},\end{aligned}$$

using that $e^{2\ln(2)}=4$ and $e^{4ik\pi}=1$. It can be written in $x+iy$ form as follows

$$(\sqrt{3}-i)^{2-i} = 4e^{-\frac{\pi}{6}-2k\pi}\cos\left(\ln(2)+\frac{\pi}{3}\right)-i4e^{-\frac{\pi}{6}-2k\pi}\sin\left(\ln(2)+\frac{\pi}{3}\right), \quad k\in\mathbb{Z}.$$

On the other hand, we have

$$\begin{aligned}\mathrm{P.V.}(\sqrt{3}-i)^{2-i} &= \exp((2-i)\,\mathrm{Log}(\sqrt{3}-i)) = \exp\left((2-i)\left(\ln(2)-i\frac{\pi}{6}\right)\right),\\ &= \exp\left(2\ln(2)-\frac{\pi}{6}-i\left(\ln(2)-\frac{\pi}{3}\right)\right),\\ &= 4e^{-\frac{\pi}{6}}\cos\left(\ln(2)-\frac{\pi}{3}\right)-i4e^{-\frac{\pi}{6}}\sin\left(\ln(2)-\frac{\pi}{3}\right).\end{aligned}$$

Exercise 2.[6 points] Evaluate the following integrals:

(1) $\displaystyle\int_0^1 t(2+it^2)^2\,\mathrm{d}t$;

(2) $\displaystyle\int_0^\pi \cos(2t+it)\,\mathrm{d}t$;

Solution.

(1) Note that $\frac{\mathrm{d}}{\mathrm{d}t}(2+it^2)^3 = 3(2+it^2)^2 \cdot 2it$ by chain rule. Hence we have

$$\begin{aligned}\int_0^1 t(2+it^2)^2\,\mathrm{d}t = \left[\frac{1}{6i}(2+it^2)^3\right]_0^1 &= \frac{(2+i)^3-2^3}{6i}\\ &= \frac{2^3+3\cdot 2^2 i+3\cdot 2\cdot i^2+i^3-2^3}{6i}\\ &= \frac{12i-6-i}{6i} = \frac{11i-6}{6i} = \frac{11}{6}+i\end{aligned}$$

(2) Note that $\cos(2t+it) = \cos((2+i)t) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2+i}\sin((2+i)t)$;

$$\int_0^\pi \cos(2t+it)\,\mathrm{d}t = \left[\frac{\sin((2+i)t)}{2+i}\right]_0^\pi = \frac{\sin((2+i)\pi)-\sin(0)}{2+i}.$$

Then we evaluate

$$\sin((2+i)\pi) = \frac{e^{i(2+i)\pi}-e^{-i(2+i)\pi}}{2i} = \frac{e^{-\pi+2i\pi}-e^{\pi-2i\pi}}{2i} = \frac{e^{-\pi}-e^{\pi}}{2i}$$

and $\frac{1}{2+i} = \frac{2-i}{2^2+1} = \frac{2-i}{5}$. Hence, we get

$$\int_0^\pi \cos(2t+it)\,\mathrm{d}t = \frac{e^{-\pi}-e^{\pi}}{2i}\cdot\frac{2-i}{5} = \frac{e^{\pi}-e^{-\pi}}{10} + i\frac{e^{\pi}-e^{-\pi}}{5}.$$

Exercise 3.[3 points] Find the zeros of the function f defined by $f(z) = \cos(iz+1)$ for $z \in \mathbb{C}$.

Solution. Recall the zeros of cos are the $k\pi + \frac{\pi}{2}$ for $k \in \mathbb{Z}$. Let $z \in \mathbb{C}$. We have

$$\begin{aligned} f(z) = 0 \quad &\Leftrightarrow \quad \cos(iz+1) = 0\\ &\Leftrightarrow \quad iz+1 = k\pi + \frac{\pi}{2} \quad \text{for some } k \in \mathbb{Z}\\ &\Leftrightarrow \quad iz = k\pi + \frac{\pi}{2} - 1 \quad \text{for some } k \in \mathbb{Z}\\ &\Leftrightarrow \quad z = i\left(-k\pi - \frac{\pi}{2} + 1\right) \quad \text{for some } k \in \mathbb{Z},\end{aligned}$$

using that $\frac{1}{i} = -i$. We conclude that the set of zeros of f is

$$\left\{ i\left(-k\pi - \frac{\pi}{2} + 1\right) : k \in \mathbb{Z}\right\}.$$

Exercise 4.[4 points] Solve the equation P.V. $z^i = -e$.

Hint: Write $\operatorname{Log} z = a + ib$ and first solve for a and b. Then recover z from a and b.

Solution. Let $z \in \mathbb{C}^*$ (note that the equation does not make sense if $z \neq 0$). Write $\operatorname{Log} z = a + ib$ with $a = \ln|z|$ and $b = \operatorname{Arg}(z)$. We have

$$\begin{aligned}
\text{P.V. } z^i = -e \quad &\Leftrightarrow \quad e^{i \operatorname{Log}(z)} = e^{1+i\pi} \\
&\Leftrightarrow \quad e^{-b+ai} = e^{1+i\pi} \\
&\Leftrightarrow \quad \begin{cases} b = -1 \\ a = \pi + 2k\pi, \quad \text{for some } k \in \mathbb{Z}, \end{cases}
\end{aligned}$$

where we use that $-1 \in (-\pi, \pi]$, so $b = -1$ is allowed (otherwise there would be no solution). Since $z = e^{a+ib}$, we get the solutions to the equation are

$$e^{\pi+2k\pi-i}, k \in \mathbb{Z}.$$

which can also be written $e^{\pi+2k\pi}\cos(1) - ie^{\pi+2k\pi}\sin(1), k \in \mathbb{Z}$.

Exercise 5.[5 points] Let I be a real interval and $w \colon I \to \mathbb{C}$ be a function. Assume w is differentiable at some $t \in I$. Prove $|w|^2$ is differentiable at t and find its derivative in terms of $w(t)$ and $w'(t)$.

Solution. We write $w(s) = u(s) + iv(s)$ for $s \in I$. Since w is differentiable at t, we know u and v are differentiable at t.

We first show that $\overline{w}$ is differentiable at t. For this, we simply note that $\overline{w(s)} = u(s) + i(-v(s))$ and u and $-v$ are differentiable at t. Hence, $\overline{w}$ is differentiable at t and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\overline{w(t)}\right) = u'(t) - iv'(t) = \overline{w'(t)}.$$

Now we recall that $|w(s)|^2 = w(s)\overline{w(s)}$ for $s \in I$. Hence by the product rule, $|w|^2$ is differentiable at t and

$$\frac{\mathrm{d}}{\mathrm{d}t}(|w(t)|^2) = w'(t)\overline{w(t)} + w(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\overline{w(t)}\right) = w'(t)\overline{w(t)} + w(t)\overline{w'(t)}.$$

This can also be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}(|w(t)|^2) = w'(t)\overline{w(t)} + \overline{w'(t)\overline{w(t)}} = 2\operatorname{Re}(w'(t)\overline{w(t)}).$$

using the formula $z + \overline{z} = 2\operatorname{Re}(z)$.

Exercise 6.[5 points] Let $\alpha \in \mathbb{R}$. Consider the branch F of the log defined by

$$F(z) = \ln r + i\theta \quad \text{for} \quad z = re^{i\theta} \quad \text{with} \quad r > 0 \quad \text{and} \quad \alpha < \theta < \alpha + 2\pi.$$

Let $D = \{re^{i\theta} : r > 0 \text{ and } \alpha < \theta < \alpha + 2\pi\}$. Recall F is analytic on D and $F'(z) = 1/z$ for any $z \in D$. Let $c \in \mathbb{C}$. We define $G(z) = e^{cF(z)}$ for any $z \in D$.

(1) Explain why G is a branch of the power function z^c on D, that is G is analytic on D and, for any $z \in D$, $G(z)$ is one of the values of z^c.
(2) For any $z \in D$, show that $G'(z) = ce^{(c-1)F(z)}$.

Remark: The principal value of the power function is only an arbitrary choice (as for the principal value of the log). Sometimes considering another one can be useful. Note that here the derivative depends on the branch chosen, which is not the case for the log.

Solution.

(1) We show G is a branch of the power function z^c on D:

- G is analytic on D by the chain rule: F is analytic on D, hence so is cF and, since exp is analytic on $\mathbb{C}$ we get that e^{cF} is analytic on D.
- Let $z \in D$. Then $F(z)$ is one of the values of $\log(z)$. But $z^c = e^{c\log(z)}$, that is the values of z^c are all the e^{cw} for w a value of $\log(z)$. Hence $G(z) = e^{cF(z)}$ is one of the values of z^c.

(2) Let $z \in D$, by the chain rule, we have

$$G'(z) = cF'(z)e^{cF(z)} = \frac{c}{z}e^{cF(z)}.$$

Now since $F(z)$ is a value of $\log(z)$, this means that $e^{F(z)} = z$ (recall $\log(z)$ is the set of complex numbers w solutions of $e^w = z$). Hence we have

$$G'(z) = \frac{c}{e^{F(z)}}e^{cF(z)} = ce^{-F(z)}e^{cF(z)} = ce^{(c-1)F(z)}.$$