Exercise 1. [6 points] Let $C$ be the arc defined by

$$z(t) = \begin{cases} \pi e^{i\pi t} & \text{if } 0 \leq t \leq 1, \\ -\pi + i(t-1) \ln(2) & \text{if } 1 \leq t \leq 2, \end{cases}$$

and $f(z) = \cos(z) \sin^2(z)$. Calculate the following integral (give your answer in $x + iy$ form)

$$\int_C f(z) \, dz.$$

Solution. $C$ is a contour because, at any $t \in [0, 1) \cup (1, 2]$, $z(t)$ is differentiable, $z'(t)$ is continuous and nonzero. The function $f$ has an antiderivative $F(z) = \frac{1}{3}(\sin(z))^3$ on the domain $C$ and the contour $C$ lies entirely in $C$ so by the theorem of Section 48,

$$\int_C f(z) \, dz = F(z(2)) - F(z(0)) = F(-\pi + i \ln(2)) - F(\pi).$$

Note that $F(\pi) = 0$ and

$$\sin(-\pi + i \ln(2)) = \frac{e^{i(\pi - i \ln(2))} - e^{-i(\pi + i \ln(2))}}{2i} = \frac{e^{-i\pi}e^{-\ln(2)} - e^{i\pi}e^{\ln(2)}}{2i} = \frac{-1 + 2}{2i} = -\frac{3i}{4}$$

so that $F(-\pi + i \ln(2)) = \frac{9i}{64}$. Finally, we get

$$\int_C f(z) \, dz = \frac{9i}{64}.$$

Exercise 2. [6 points] Let $z_0 \in \mathbb{C}$ and $r > 0$. Let $C$ be the positively oriented circle of radius $r$ about $z_0$ given by

$$z(\theta) = z_0 + re^{i\theta}, \quad 0 \leq \theta \leq 2\pi.$$

Evaluate the following integral (give your answer in terms of $z_0$)

$$\int_C \frac{z + i}{z - z_0} \, dz.$$

Solution. Using the definition of contour integrals

$$\int_C \frac{z + i}{z - z_0} \, dz = \int_0^{2\pi} \frac{z_0 + re^{i\theta} + i}{z_0 + re^{i\theta} - z_0} rie^{i\theta} \, d\theta$$

$$= \int_0^{2\pi} \frac{z_0 + i + re^{i\theta}}{re^{i\theta}} rie^{i\theta} \, d\theta$$

$$= \int_0^{2\pi} (iz_0 - 1 + ire^{i\theta}) \, d\theta$$

$$= 2\pi(iz_0 - 1) + [re^{i\theta}]_0^{2\pi}$$

$$= 2\pi(iz_0 - 1).$$
Alternative approach: With Cauchy integral formula applied with \( f(z) = z + i \) which is analytic on and within \( C \), we have

\[
\int_C \frac{z + i}{z - z_0} \, dz = 2i\pi f(z_0) = 2i\pi(z_0 + i) = 2\pi(iz_0 - 1).
\]

Exercise 3. [6 points] Let \( C \) be a closed contour. Let \( f \) be a piecewise continuous function on \( C \). Prove that the integral \( \int_C f(z) \, dz \) does not depend of the choice of the initial point of the contour. More precisely, assume \( C \) is given by \( z = z(t) \), \( a \leq t \leq b \), fix some \( t_0 \in [a, b] \) and define \( C' \) by

\[
z = w(t) = \begin{cases} z(t) & \text{if } t_0 \leq t \leq b, \\ z(t - b + a) & \text{if } b \leq t \leq b - a + t_0, \end{cases}
\]

Then you have to prove \( \int_C f(z) \, dz = \int_{C'} f(z) \, dz \).

Solution. Using the definition of contour integrals

\[
\int_{C'} f(z) \, dz = \int_{t_0}^{b-a+t_0} f(w(t))w'(t) \, dt
\]

\[
= \int_{t_0}^{b} f(z(t))z'(t) \, dt + \int_{b}^{b-a+t_0} f(z(t - b + a))z'(t - b + a) \, dt.
\]

In the second term we use the change of variable \( s = t - b + a \), noting that when \( t \) goes from \( b \) to \( b - a + t_0 \), \( s \) goes from \( a \) to \( t_0 \). Hence, we get

\[
\int_C f(z) \, dz = \int_{t_0}^{b} f(z(t))z'(t) \, dt + \int_{a}^{t_0} f(z(s))z'(s) \, ds = \int_{a}^{b} f(z(t))z'(t) \, dt = \int_C f(z) \, dz,
\]

where we replaced \( s \) by \( t \) (it is just a dummy variable) and then combined both integrals.

Exercise 4. [6 points] Let \( C \) be the arc defined by

\[
z(t) = \begin{cases} it & \text{if } 0 \leq t \leq 1, \\ i(t - 1) & \text{if } 1 \leq t \leq 2, \\ 1 + i(t - 2) & \text{if } 2 \leq t \leq 3, \\ 1 - (t - 3) & \text{if } 3 \leq t \leq 4. \end{cases}
\]

Evaluate the following integral (give your answer in \( x + iy \) form)

\[
\int_C \frac{e^{z^2}}{z^2 + 4} \, dz
\]

Solution. First note that \( C \) is a contour because, at any \( t \in [0, 1) \cup (1, 2) \cup (2, 3) \cup (3, 4] \), \( z(t) \) is differentiable, \( z'(t) \) is continuous and nonzero. Moreover, note that \( C \) is the square with vertices 0, \( i \), 1 + \( i \), 1, which is a simple closed contour. But \( \frac{e^{z^2}}{z^2 + 4} \) is analytic everywhere except when \( z^2 + 4 = 0 \) that is when \( z = \pm 2i \), which are not on or within \( C \). So by Cauchy-Goursat theorem,

\[
\int_C \frac{e^{z^2}}{z^2 + 4} \, dz = 0.
\]

Exercise 5. [6 points] Let \( C \) be the following contour (its exact definition does not matter but some of its properties do):
Let $f(z) = \text{P.V.} \, z^{1/3}$ for $z \neq 0$. Evaluate the following integral (give your answer in $x + iy$ form)
\[ \int_C f(z) \, dz. \]

**Solution.** Let $F(z) = \frac{3}{4} \text{P.V.} \, z^{4/3}$ for $z \neq 0$. This function is analytic on $\mathbb{C} \setminus \mathbb{R}^-$ and we have seen that its derivative is, for any $z \in \mathbb{C} \setminus \mathbb{R}^-$,
\[ F'(z) = \frac{3}{4} \cdot \frac{4}{3} \text{P.V.} \, z^{(4/3)-1} = \text{P.V.} \, z^{1/3} = f(z). \]
Therefore, $f$ has an antiderivative on $\mathbb{C} \setminus \mathbb{R}^-$. But the contour $C$ is included in $\mathbb{C} \setminus \mathbb{R}^-$, so by the theorem of Section 48,
\[ \int_C f(z) \, dz = F(-i) - F(-2 + 2i), \]
since $-2 + 2i$ is the initial point of $C$ and $-i$ the final point. With $-i = e^{-i\pi/2}$, we get
\[ F(-i) = \frac{3}{4} \exp \left( \frac{4}{3} \text{Log}(-i) \right) = \frac{3}{4} \exp \left( \frac{4}{3} \cdot \left( -\frac{i\pi}{2} \right) \right) = \frac{3}{4} \exp \left( -\frac{2i\pi}{3} \right) = \frac{3}{8} - \frac{3\sqrt{3}}{8}i. \]
With $-2 + 2i = 2^{3/2}e^{3i\pi/4}$, we get
\[ F(-2 + 2i) = \frac{3}{4} \exp \left( \frac{4}{3} \text{Log}(-2 + 2i) \right) = \frac{3}{4} \exp \left( \frac{4}{3} \cdot \left( \ln(2^{3/2}) + \frac{3i\pi}{4} \right) \right) = \frac{3}{4} \exp \left( 2\ln(2) + i\pi \right) = \frac{3}{4} \cdot 2^2 e^{i\pi} = -3. \]
So finally we get
\[ \int_C f(z) \, dz = \frac{3}{8} - \frac{3\sqrt{3}}{8}i + 3 = \frac{21}{8} - \frac{3\sqrt{3}}{8}i. \]