Dynamical Systems, homework 1.

Exercise 1. Let $f(x) = x^3 - 2x^2 + x$. Let $(x_n)_{n\geq 0}$ be defined by iterations of the Newton method for f, beginning from $x_0 \in \mathbb{R}$.

a) What are the fixed points of the Newton method?

b) Prove that there is a neighborhood A of 0 such that for any $x_0 \in A$ and $\lambda \in (1, 2)$ there are $a > 0$, $b \in (0, 1)$ such that for any $n \geq 0$

$$|x_n| < a\, b^{\lambda^n}.$$

b) Prove that there is a neighborhood B of 1 such that for any $x_0 \in B$, $x_0 \neq 1$, and $\mu \in (1/2, 1)$ there is $c > 0$, such that for any $n \geq 0$

$$|x_n - 1| < c\, \mu^n.$$

c) By writing $x_{n+1} = F(x_n)$ and calculating $F'(0)$ and $F'(1)$ (obtained by continuous extension), justify the difference of the speed of convergence in the two previous questions.

d) What are the basins of attraction of 0 and 1?

Exercise 2. Consider the function $P(x) = x^4 - x^2 - 11/36$.

a) Compute the inflection points of P. Show that they are critical points for the associated Newton function.

b) Prove that these two points lie on a 2-cycle.

c) What can you say for the convergence of Newton's method for this function?

Exercise 3. Let

$$T(x) = \begin{cases} 2x & for\ x \leq 1/2 \\ 2 - 2x & for\ x \geq 1/2 \end{cases}$$

be the *tent* map.

a) Sketch the graph on $I = [0, 1]$ of T, $T^{\circ 2}$, and a representative graph of $T^{\circ n}$ for $n > 2$.

b) Use the graph of $T^{\circ n}$ to conclude that T has exactly 2^n points of period n.

c) Prove that the set of all periodic points of T is dense in I.

Exercise 4. Find the fixed points of the cubic Henon map,

$$T(x, y) = (cx - x^3 - y, x),$$

and analyze their stability, depending on the parameter c.

Exercise 5. Let $n \in \mathbb{N}^*$ and $f : S^1 \to S^1$ be defined by $f(\theta) = \theta + \epsilon \sin(n\theta)$, for $0 < \epsilon < 1/n$. What are the fixed points? Identify the attracting and repelling ones.

Exercise 6. Let f be a diffeomorphism in $\mathbb{R}$. Prove that all hyperbolic periodic points are isolated[1].

[1] A point with primitive period n is said to be hyperbolic if $|f^{\circ n}(x)| \neq 1$

Exercise 7. Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial map. How many periodic points of period n does it have, where we count the periodic points with multiplicity and do not require the periodic points to be of minimal period?

Exercise 8. Let $f : \mathbb{R} \to \mathbb{R}$ be smooth, and p a fixed point with

$$f'(p) = 1,\ \ f''(p) > 0.$$

What can you say about the convergence of $(f^{\circ n}(x_0), n \geq 1)$, for x_0 in a neighborhood of p?

Exercise 9. Suppose that P and Q are polynomials, and let $F = P/Q$. What can be said about the associated Newton method for F? Which fixed points are attracting and which are repelling?

Exercise 10. Prove that a homeomorphism of $\mathbb{R}$ cannot have periodic points with prime period greater than 2.