Dynamical Systems, homework 3.

Exercise 1. Describe the dynamics of the linear maps given by the following matrices:

$$\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}.$$

Exercise 2. Calculate the Lyapunov exponent, at a given point x, for the linear maps given by the following matrices:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1/2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

What are the fixed points, and are they stable?

Exercise 3. Assume $f : X \to X$ and $g : Y \to Y$ are topologically conjugate via $h : X \to Y$ (here X and Y are complete metric spaces and h is an homeomorphism). Show the following assertions.

a) If p is a periodic point of f of prime period n, $h(p)$ is a periodic point of g of prime period n. If p is eventually periodic for f, $h(p)$ is eventually periodic for g.

b) If p is an attracting periodic point[1] for f with stable set $W(p)$, then $h(p)$ is attracting periodic for g with stable set $h(W(p))$.

c) If p is a repelling periodic point[2] for f, then $h(p)$ is a repelling periodic point for g.

d) If f has a dense orbit, g has a dense orbit.

Exercise 4. A continuous function $f : [0, 1] \to [0, 1]$ is called unimodal if:

(1) f has a unique maximum point $c \in (0, 1)$ with $f(c) = 1$;
(2) f is strictly increasing on $[0, c]$ and strictly decreasing on $[c, 1]$;
(3) $f(0) = f(1) = 0$.

Prove that such a function has a least 2^n periodic points with prime period n in $[0, 1]$.

Exercise 5. Prove that an orientation reversing diffeomorphism of the circle must have two fixed points.

Exercise 6. Assume that f is a smooth function such that $S(f) < 0$ for any $x \in \mathbb{R}$, p is a one-sided attracting periodic point, and $W(p)$ is the maximal stable interval containing p. Assume that $W(p)$ is bounded. Show that for some i, there is a critical point of f in $W(f^{\circ i}(p))$

[1] This means that for some $n \in \mathbb{N}$, prime period of p, and some open neighborhood X of p, $f^{\circ kn}(x) \to p$ as $k \to \infty$ for any $x \in X$.

[2] This means that for some $n \in \mathbb{N}$, prime period of p, and some $\epsilon > 0$, in any open neighborhood X of p there exists an $x \in X$ and $k \in \mathbb{N}$ such that $\mathrm{d}(f^{\circ kn}(x), p) > \epsilon$.