Dynamical Systems, homework 4.

Exercise 1.

(a) Compute the Hausdorff dimension of the Menger sponge. This three dimensional set can be obtained iteratively as the Cantor set by successively taking away the middle rectangular columns of each complete cube. See e.g.

en.wikipedia.org/wiki/Menger_sponge.

(b) Give a criterion for a point in $[0,1]^3$ to be part of the Menger sponge.

Exercise 2. Show that the differential equation $\dot{x} = 3x^{2/3}$, $x(0) = 0$, has infinitely many solutions given by

$$x(t) = \begin{cases} 0 & \text{if } t < c \\ (t-c)^3 & \text{if } t \geq c \end{cases}.$$

Check that the function $x^{2/3}$ is not Lipschitz.

Exercise 3. Assume that e^{tA} and e^{tB} are contractions on $\mathbb{R}^n$. Prove that if $AB = BA$, then $e^{t(A+B)}$ is also a contraction. What if $AB \neq BA$?

Exercise 4. Consider the non-homogeneous linear system $\dot{x} = Ax + b(t)$, where $x \in \mathbb{R}^n$ and $b : \mathbb{R} \to \mathbb{R}^n$.

(a) Find a particular solution of type $x(t) = e^{tA}z(t)$ for an appropriate $z : \mathbb{R} \to \mathbb{R}^n$. This is called the method of *variation of constants*. Show that the general solution with initial conditions $x(0) = c$ has the form

$$x(t) = \int_0^t e^{(t-s)A} b(s) \mathrm{d}s + e^{tA}c.$$

(b) Use the previous question to solve

$$\dot{x} = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix} x + \begin{pmatrix} \sin(3t) \\ \cos(3t) \end{pmatrix}.$$

Exercise 5. Apply the Picard iteration to solve $\dot{x} = 1 + x^2$ with $x(0) = 0$. Show that after three Picard iterations, the result agrees with the true solution for terms of degree five or less in t. Can the solution to this equation be continued for all time?

Exercise 6. Consider the solution $x_0 e^{tA}$ of the linear differential equation $\dot{x} = Ax$, with $x \in \mathbb{R}^n$.

(a) Prove that all the eigenvalues of A have nonzero real part (i.e., the fixed point 0 is hyperbolic) if and only if for any $x_0 \in \mathbb{R}^n$, either $\omega(x_0) = \{0\}$ or $\omega(x_0) = \varnothing$.

(b) For $n = 4$, show that there is a choice of A and a point x_0 such that $\mathcal{O}(x_0) \subset \omega(x_0)$ but $\mathcal{O}(x_0)$ is neither a fixed point nor a periodic orbit.

Exercise 7. Consider the differential equations

$$\begin{cases} \dot{x} &= a - x - \frac{4xy}{1+x^2} \\ \dot{y} &= b\,x\left(1 - \frac{y}{1+x^2}\right) \end{cases}$$

for $a, b > 0$.

(a) Show that $x^* = a/5, y^* = 1 + (x^*)^2$, is the only fixed point.

(b) Show that the fixed point is repelling for $b < 3a/5 - 25/a$ and $a > 0$. Hint: show that the determinant and trace of the Jacobian matrix are positive at (x^*, y^*).

(c) Let x_1 be the value of x where the line $\{\dot{x} = 0\}$ crosses the x-axis. Let $y_1 = 1 + x_1^2$. Prove that the rectangle $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq x_1,\, 0 \leq y \leq y_1\}$ is positively invariant.

(d) Prove that there is a periodic orbit in the first quadrant for $a > 0$ and $0 < b < 3a/5 - 25/a$.

Exercise 8. Discuss the basin of attraction of the fixed points for the following system of differential equations with $\delta > 0$:

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= x - x^3 - \delta\, y \end{cases}$$

Exercise 9.

(a) Verify Dulac's criterion: assuming $\dot{x} = V(x)$ is a differential equation in a region D of the plane, if there exists a smooth function f such that $\mathrm{div}(fV)$ has no zeros in D, then there are no cycles in D.

(b) Use Dulac's criterion to prove that

$$\begin{cases} \dot{x} &= x(2 - x - y) \\ \dot{y} &= y(4x - x^2 - 3) \end{cases}$$

has no closed cycles in $D = \{x > 0, y > 0\}$. Hint: try $f(x, y) = 1/(xy)$.

Exercise 10. Let $A = S^1 \times [a, b]$ be the an annulus with covering space $\tilde{A} = \mathbb{R} \times [a, b]$ (the angle variable is not taken modulo 1 in the covering space). Let X be a vector field on A with lift $\tilde{X} = (\tilde{X}_1, \tilde{X}_2)$ on $\tilde{A}$. Assume that $\tilde{X}_1(x, a) < 0$ and $\tilde{X}_1(x, b) > 0$ for all x, and $\mathrm{div}(X) = 0$, i.e. X is area preserving. Prove that the flow of X has a fixed point in A.

Hint: Assume it has no fixed point. and let Y be a nonzero vector field which is perpendicular to X everywhere in A. prove that Y has a periodic orbit γ in A using the Poincaré bendixson theorem. Get a contradiction to the area preserving assumption by considering X along γ.