Probability, homework 2.

Exercise 1. Write down a precise and complete proof of the following Monotone Class Theorem and its corollary.

- Let $\mathcal{C}$ be a class of subsets of Ω, closed under finite intersections and containing Ω. Let $\mathcal{B}$ be the smallest class containing $\mathcal{C}$ which is closed under increasing limits and by difference. Then $\mathcal{B} = \sigma(\mathcal{C})$
- Let P and Q be two probabilities defined on $\mathcal{A}$, and suppose that they agree on a class $\mathcal{C} \subset \mathcal{A}$ which is closed under finite intersections. If $\sigma(\mathcal{C}) = \mathcal{A}$, we have $P = Q$.

Exercise 2. The probability that a male driver makes an insurance claim in any given year is 0.3, while the probability that a female driver makes an insurance claim in any given year is 0.2. Furthermore, claims by the same driver in successive years are independent events. We assume equal numbers of male and female drivers.

What is the probability that a randomly chosen driver makes a claim in the first year (event A)? What is the probability that a randomly chosen driver makes a claim in the first and second years (event B)?

What is $\mathbb{P}(B \mid A)$, the probability that a randomly chosen driver makes a claim in the second year, conditionally to the fact that he/she made one on the first year? How can you explain that it is different from $\mathbb{P}(A)$ although claims in successive years are independent? If you are the head of an insurance company and want one more client, would you prefer one who had a claim the previous year or the contrary?

Exercise 3. Let X be a geometric random variable. Prove the following memoryless property: for $i, j > 0$,

$$\mathbb{P}(X > i + j \mid X \geq i) = \mathbb{P}(X > j).$$

Exercise 4. A simple given property is genetically encoded as a pair of alleles a and A, which yields three possible genotypes $\{A, A\})$, $\{a, a\}$, and $\{A, a\}$, represented in the population with respective probabilities p, q, r, $p + q + r = 1$, homogenously and with these same probabilities for each gender. A parent passes on one of its alleles, chosen at random uniformly, to its child; the genotype of the child combines alleles from both parents.

We assume that these coefficients p, q, r are stable from one generation to another. Explain why they actually depend on only one parameter:

$$\begin{cases} p = P^2, \\ q = Q^2, \\ r = 2PQ, \end{cases}$$

where $P + Q = 1$.

Exercise 5. Let $c > 0$ and X be a real random variable such that for any $\lambda \in \mathbb{R}$

$$\mathbb{E}\left(e^{\lambda X}\right) \leq e^{c\frac{\lambda^2}{4}}.$$

Prove that, for any $\delta > 0$,

$$\mathbb{P}\left(|X| > \delta\right) \leq 2e^{-\frac{\delta^2}{c}}.$$

Exercise 6. Let $(s_n)_{n\geq 0}$ be a 1-dimensional, unbiased random walk. For $a, b \in \mathbb{Z}$, let $T_a = \inf\{n \geq 0 : s_n = a\}$ and $T_{a,b} = \inf\{n \geq 0 : s_n = a \text{ or } s_n = b\}$. For $x \in \mathbb{Z}$, let $\omega(x) = \mathbb{P}(s_{T_{a,b}} = b \mid s_0 = x)$.

Prove that for $a < x < b$, $\omega(x) = \frac{1}{2}(\omega(x+1) + \omega(x-1))$, provided we define $\omega(a) = 0$ and $\omega(b) = 1$. Conclude that

$$\omega(x) = \frac{x-a}{b-a}.$$

From this result, how can you recover that $\mathbb{P}(T_b < \infty) = 1$?