Probability, homework 4.

Exercise 1.

(a) Let $(f_n)_{n\geq 0}$ be a sequence of nonnegative functions converging almost surely (for the Lebesgue measure $\mathrm{d}\mu$) to f. Assume that $\int f_n \mathrm{d}\mu \to c < \infty$ as $n \to \infty$. Prove that $\int f \mathrm{d}\mu$ is defined in $[0, c]$, but does not have to be necessarily c.

(b) Build a sequence of functions $(f_n)_{n\geq 0}$, $0 \leq f_n \leq 1$, such that $\int f_n \mathrm{d}\mu \to 0$ but for any $x \in \mathbb{R}$, $(f_n(x))_{n\geq 0}$ does not converge.

Exercise 2. Let $(d_n)_{n\geq 0}$ be a sequence in $(0, 1)$, and $K_0 = [0, 1]$. We define iteratively $(K_n)_{n\geq 0}$ in the following way. From K_n, which is the union of closed disjoint intervals, we define K_{n+1} by removing from each interval of K_n an open interval, centered at the middle of the previous one, with length d_n times the length of the previous one. Let $K = \cap_{n\geq 0} K_n$ (K is called a Cantor set).

(a) Prove that K is an uncountable compact set, with empty interior, and whose points are all accumulation points

(b) What is the Lebesgue measure of K?

Exercise 3. On a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is given a random variable (X, Y) with values in $\mathbb{R}^2$.

(a) If the law of (X, Y) is $\lambda\mu e^{-\lambda x - \mu y} \mathbb{1}_{\mathbb{R}_+^2}(x, y)\mathrm{d}x\mathrm{d}y$, what is the law of $\min(X, Y)$?

(b) If the law of (X, Y) is $\frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} \mathrm{d}x\mathrm{d}y$, what is the law of X/Y?

Exercise 4. Let $\alpha > 0$ and, given $(\Omega, \mathcal{A}, \mathbb{P})$, let $(X_n, n \geq 1)$ be a sequence of independent real random variables with law $\mathbb{P}(X_n = 1) = \frac{1}{n^\alpha}$ and $\mathbb{P}(X_n = 0) = 1 - \frac{1}{n^\alpha}$. Prove that $X_n \to 0$ in $\mathcal{L}^1$, but that almost surely

$$\limsup_{n\to\infty} X_n = \begin{cases} 1 & \text{if} \quad \alpha \leq 1 \\ 0 & \text{if} \quad \alpha > 1 \end{cases}.$$

Exercise 5. You toss a coin repeatedly and independently. The probability to get a head is p, a tail is $1-p$. Let A_k be the following event: k or more consecutive heads occur amongst the tosses numbered $2^k, \dots, 2^{k+1} - 1$. Prove that $\mathbb{P}(A_k \text{ i.o.}) = 1$ if $p \geq 1/2$, 0 otherwise.

Exercise 6. Let $\epsilon > 0$ and X be uniformly distributed on $[0, 1]$. Prove that, almost surely, there exists only a finite number of rationals $\frac{p}{q}$, with $p \wedge q = 1$, such that

$$\left| X - \frac{p}{q} \right| < \frac{1}{q^{2+\epsilon}}.$$