Probability, homework 7.

Exercise 1. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with standard Cauchy distribution, on the same probability space, and let $M_n = \max(X_1,\dots,X_n)$. Prove that $(nM_n^{-1})_{n\geq 1}$ converges in distribution and identify the limit.

Exercise 2. *Convergence in L^1 in the strong law of large numbers.*

a) Prove that if S_n converges to S almost surely, and $(S_n)_{n\geq 1}$ is uniformly integrable, then S_n converges to S in L^1.

b) Prove that if the X_ℓ's are i.i.d. and in L^1, then $(n^{-1}\sum_{k=1}^n X_k)_{n\geq 1}$ is uniformly integrable.

c) Conclude that the strong law of large numbers in the almost sure sense implies the strong law of large numbers in the L^1 sense.

Exercise 3. Let $(X_n)_{n\geq 1}$ be a sequence of random variables, on the same probability space, with $\mathbb{E}(X_\ell) = \mu$ for any ℓ, and a weak correlation in the following sense: $\mathrm{Cov}(X_k, X_\ell) \leq f(|k-\ell|)$ for all indexes k, ℓ, where the sequence $(f(m))_{m\geq 0}$ converges to 0 as $m\to\infty$. Prove that $(n^{-1}\sum_{k=1}^n X_k)_{n\geq 1}$ converges to μ in L^2.

Exercise 4 Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables, on the same probability space, with law given by $\mathbb{P}(X_1 = (-1)^m m) = 1/(cm^2\log m)$ for $m\geq 2$ (c is the normalization constant $c = \sum_{m\geq 2} 1/(m^2\log m)$). Prove that $\mathbb{E}(|X_1|) = \infty$, but there exists a constant $\mu \notin \{\pm\infty\}$ such that $(n^{-1}\sum_{k=1}^n X_k)_{n\geq 1}$ converges to μ in probability. Does it converge almost surely, and in L^p?

Exercise 5. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. Bernoulli random variables, on the same probability space, with parameter $1/2$, and let τ_n be the hitting time of level n by the partial sums, i.e. $\tau_n = \inf\{k \mid \sum_{\ell=1}^k X_\ell = n\}$. Show that $n^{-1}\tau_n$ converges to 2 almost surely.

Exercise 6. *Kolmororov's maximal inequality and convergence of random series.* Let $(X_n)_{n\geq 1}$ be a sequence of mutually independent random variables, on the same probability space, with expectation 0 and finite variance. Let $S_n = \sum_{\ell=1}^n X_\ell$. Prove that for any $\lambda > 0$,

$$\lambda^2\,\mathbb{P}(\max_{1\leq k\leq n} |S_k| \geq \lambda) \leq \mathrm{Var}(S_n).$$

Prove that if $\sum_\ell \mathrm{Var}(X_\ell) < \infty$, then $(S_n)_{n\geq 1}$ converges almost surely.