Probability, homework 1.

Exercise 1.

(i) A family has 5 children, consisting of 3 girls and 2 boys. Assuming that all birth orders are equally likely, what is the probability that the 3 eldest children are girls?
(ii) How many ways are there to split 11 people into 3 teams, where one team has 2 people, one has 4 and the other 5?

Exercise 2.

(i) You are n students attending the probability course. Assume all of your birth days are independent, all days of the year have the same probability, and all years have 365 days. What is the probability that at least two of you celebrate their birthday the same day?
(ii) Evaluate numerically this probability for $n = 20$, 30, 40.

Exercise 3. Assume you attend a college in which each class meets only once a week. You decide between 30 non-overlapping classes. There are 6 classes to choose from for each day of the week, Monday through Friday. Trusting randomness, you decides to register for 7 randomly selected classes out of the 30, with all choices equally likely. What is the probability that you will have classes every day, Monday through Friday?

Exercise 4. Prove whether the following sets are countable or not.

(i) All intervals in $\mathbb{R}$ with rational endpoints.
(ii) All circles in the plane with rational radii and centers on the diagonal $x = y$.
(iii) All sequences of integers whose terms are either 0 or 1.

Exercise 5. Let $\varnothing \subsetneq A \subsetneq B \subsetneq \Omega$ (these are strict inclusions). What is the σ-algebra generated by $\{A, B\}$?

Exercise 6. The symmetric difference of two events A and B, denoted $A \triangle B$, is the event that precisely one of them occurs: $A \triangle B = (A \cup B) \backslash (A \cap B)$.

(i) Write a formula for $A \triangle B$ that only involves the operations of union, intersection and complement, but no set difference.
(ii) Define $d(A, B) = \mathbb{P}(A \triangle B)$. Show that for any three events A, B, C,

$$d(A, B) + d(B, C) - d(A, C) = 2 \left(\mathbb{P} \left(A \cap B^c \cap C \right) + \mathbb{P} \left(A^c \cap B \cap C^c \right) \right).$$

(iii) Assume $A \subset B \subset C$. Prove that $d(A, C) = d(A, B) + d(B, C)$.