Probability, homework 2, due September 24.

Exercise 1. Suppose that Ω is an infinite set (countable or not), and let $\mathcal{A}$ be the family of all subsets which are either finite or have finite complement. Prove that $\mathcal{A}$ is not a σ-algebra.

Exercise 2. Let $\mathcal{A}$ be a σ-algebra, $\mathbb{P}$ a probability measure and $(A_n)_{n\geq 1}$ a sequence of events in $\mathcal{A}$ which converges to A. Prove that

(i) $A \in \mathcal{A}$;
(ii) $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A)$.

Exercise 3. Let $\mathcal{A}$ be a σ-algebra, $\mathbb{P}$ a probability measure and $(A_n)_{n\geq 1}$ (resp. $(B_n)_{n\geq 1}$) be a sequence of events in $\mathcal{A}$ which converges to A (resp. B). Assume that $\mathbb{P}(B) > 0$ and $\mathbb{P}(B_n) > 0$ for all n. Show that

(i) $\lim_{n\to\infty} \mathbb{P}(A_n \mid B) = \mathbb{P}(A \mid B)$;
(ii) $\lim_{n\to\infty} \mathbb{P}(A \mid B_n) = \mathbb{P}(A \mid B)$;
(iii) $\lim_{n\to\infty} \mathbb{P}(A_n \mid B_n) = \mathbb{P}(A \mid B)$.

Exercise 4. Let A, B, C be three mutually independent events and $\mathbb{P}(B \cap C) \neq 0$. Prove that $\mathbb{P}(A \mid B \cap C) = \mathbb{P}(A)$.

Exercise 5. A pair of dice is rolled until a sum of either 5 or 7 appears. Find the probability that a 5 occurs first. Hint: consider the event E_n that a 5 occurs on the nth roll and no 5 or 7 occurs on the first $(n-1)$ rolls.

Exercise 6. Suppose that 5 percent of men and 2 percent of women are color blind. A color-blind person is chosen at random. What is the probability of this person being male? Assume that there are an equal number of males and females. What if the population consisted of twice as many males as females?

Exercise 7 (bonus). A deck of 52 cards (26 red and 26 black) is shuffled so that all orderings are equally likely. We then play the following game: I turn the cards over one at a time so that you can see them. Before I have turned all cards, at some point you say "stop". At this point I turn the next card and if it is red I give you one dollar, if black you give me one dollar. Propose a strategy to maximize the expectation of your gain.