Probability, homework 4, due October 15.

Exercise 1. Let X be uniformly distributed on $[0,1]$ and $\lambda > 0$. Show that $-\lambda^{-1}\log X$ has the same distribution as an exponential random variable with parameter λ.

Exercise 2. Let $X_1,\dots,X_n$ be bounded, independent and identically distributed random variables such that $\mathbb{E}(X_1)=0$, $\mathbb{E}(X_1^2)=\sigma^2$, $\mathbb{E}(X_1^4)=\kappa^4$.

(i) Calculate $\mathbb{E}\left((\sum_{k=1}^n X_i)^4\right)$.
(ii) Prove that for any $\varepsilon > 0$ and any random variable X, $\mathbb{P}(|X| > \varepsilon) \le \varepsilon^{-4}\,\mathbb{E}(X^4)$.
(iii) Conclude that $\frac{X_1+\dots+X_n}{n}$ converges to 0 almost surely, as $n \to \infty$.
(iv) Explain why, in the above proof of a law of large numbers, the second moment (instead of fourth) would not be sufficient.

Exercise 3. Let X, Y be random variables such that X, Y ad XY are in L^1. Assume $\mathbb{E}(XY)=\mathbb{E}(X)\,\mathbb{E}(Y)$. By giving an example, prove that X and Y are not necessarily independent.

Exercise 4. Let X and Y be real random variables such that $\mathbb{E}(X^2),\mathbb{E}(Y^2) < \infty$. Prove that, if X and Y are independent, then

$$\mathrm{Cov}(X,Y) := \mathbb{E}((X-\mathbb{E}(X))(Y-\mathbb{E}(Y))) = 0.$$

Exercise 5. Let X, Y be in L^1. Prove that, if X and Y are independent, $XY \in \mathrm{L}^1$. Show this is not true in general (i.e. if X and Y are not independent).

Exercise 6. Let X, Y be independent random variables with positive integers values, with distribution

$$\mathbb{P}(X=i)=\mathbb{P}(Y=i)=\frac{1}{2^i}, i \in \mathbb{N}^*.$$

Find the following proabilitities.

(i) $\mathbb{P}(\max(X,Y) \ge i)$.
(ii) $\mathbb{P}(X=Y)$.
(iii) $\mathbb{P}(X > Y)$.
(iv) $P(X \text{ divides } Y)$.

Exercise 7. Let X be a geometric random variable (i.e. X has vales in $\mathbb{N}$ and $\mathbb{P}(X=i)=(1-p)^i p$ for some fixed $p \in (0,1)$). Prove the following memoryless property: for $i, j > 0$,

$$\mathbb{P}(X > i+j \mid X \ge i) = \mathbb{P}(X > j).$$

Exercise 8. Let X be a standard Gaussian random variable. What is the density of $1/X^2$?

Exercise 9. In the (O,x,y) plane, a random ray emerges from a light source at point $(-1,0)$, towards the (O,y) axis. The angle with the (O,x) axis is uniform on $(-\frac{\pi}{2},\frac{\pi}{2})$. What is the distribution of the impact point with the (O,y) axis?

Exercise 10. Let $\alpha > 0$ and, given $(\Omega,\mathcal{A},\mathbb{P})$, let $(X_n, n \geq 1)$ be a sequence of independent real random variables with law $\mathbb{P}(X_n = 1) = \frac{1}{n^\alpha}$ and $\mathbb{P}(X_n = 0) = 1 - \frac{1}{n^\alpha}$. Prove that $X_n \to 0$ in $\mathcal{L}^1$, but that almost surely

$$\limsup_{n\to\infty} X_n = \begin{cases} 1 & \text{if} \quad \alpha \leq 1 \\ 0 & \text{if} \quad \alpha > 1 \end{cases} .$$

Exercise 11 (Bonus). Let $(s_n)_{n\geq 0}$ be a 1-dimensional, unbiased random walk. For $a,b \in \mathbb{Z}$, let $T_a = \inf\{n \geq 0 : s_n = a\}$ and $T_{a,b} = \inf\{n \geq 0 : s_n = a \textit{ or } s_n = b\}$. For $x \in \mathbb{Z}$, let $\omega(x) = \mathbb{P}(s_{T_{a,b}} = b \mid s_0 = x)$.

Prove that for $a < x < b$, $\omega(x) = \frac{1}{2}(\omega(x+1) + \omega(x-1))$, provided we define $\omega(a) = 0$ and $\omega(b) = 1$. Conclude that

$$\omega(x) = \frac{x-a}{b-a}.$$

From this result, prove that $\mathbb{P}(T_b < \infty) = 1$.