Probability, homework 5, due October 22.

Exercise 1. Let X be uniform on $(-\pi,\pi)$ and $Y=\sin(X)$. Show that the density of Y is
$$\frac{1}{\pi\sqrt{1-y^2}}\mathbb{1}_{[-1,1]}(y).$$

Exercise 2. Let (X,Y) be uniform on the unit ball, i.e. it has density
$$f_{(X,Y)}(x,y)=\left\{\begin{array}{lll}\frac{1}{\pi} & \text{if} & x^2+y^2\leq 1,\\ 0 & \text{if} & x^2+y^2>1.\end{array}\right.$$
Find the density of $\sqrt{X^2+Y^2}$.

Exercise 3. Let X be a random variable with density $f_X(x)=(1-|x|)\mathbb{1}_{(-1,1)}(x)$. Show that its characteristic function is
$$\phi_X(u)=\frac{2(1-\cos u)}{u^2}.$$

Exercise 4. Let (X,Y) have density $\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}$. What is the density of X/Y?

Exercise 5. Let $(d_n)_{n\geq 0}$ be a sequence of real numbers in $(0,1)$, and $K_0=[0,1]$. We define the sequence $(K_n)_{n\geq 0}$ as follows: from K_n (a union a closed disjoint intervals) we define K_{n+1} by removing in each interval I from K_n an open interval, centered at the middle of I, with length d_n times the length of I. Let $K=\cap_{n\geq 0}K_n$.

(1) Prove that K is compact, uncountable, containing no open interval.
(2) What is the Lebesgue measure of K?

Exercise 6. A monkey tries to do some typesetting, by successively pushing one of the 84 keyboard buttons, forever, each button being chosen uniformly and independently of the others. Prove that almost surely, after some time, he will type the exact sequence of letters in James Joyce's Ulysses.

Exercise 7 (Bonus). Let $\varepsilon>0$ and X be uniformly distributed on $[0,1]$. Prove that, almost surely, there exists only a finite number of rationals $\frac{p}{q}$, with $p\wedge q=1$, such that
$$\left|X-\frac{p}{q}\right|<\frac{1}{q^{2+\varepsilon}}.$$