Probability, final

Grade will be scaled so that six exercises perfectly solved will give maximal score 100/100. Any types of notes, books and calculators are forbidden.

Exercise 1. Ten people throw their hats in a box and then randomly redistribute among themselves (each person getting one hat, all 10! permutations being equally likely). Let N be the number of people who get their own hat back. Compute the following:

(i) $\mathbb{E}(N)$;
(ii) $\mathbb{P}(N=9)$;
(iii) $\mathbb{P}(N=8)$.

Exercise 2 Consider a routine screening test for a disease. Suppose the frequency of the disease in the population (base rate) is 0.5%. The test is accurate with a 3% false positive rate (i.e. $\mathbb{P}(\text{positive} \mid \text{not sick}) = 3\%$) and a 8% false negative rate (i.e. $\mathbb{P}(\text{negative} \mid \text{sick}) = 8\%$). You take the test and it comes back positive. What is the probability that you have the disease?

Exercise 3 Let X be distributed as a standard Cauchy random variable. What is the density of X^2?

Exercise 4. Let X and Y be two independent exponential random variables with parameter 1. What is the density of $X+Y$?

Exercise 5 Let the X_ℓ's be i.i.d. real random variables, uniform on $[0,1]$. What is the limit of $(X_1^3+\cdots+X_n^3)/(X_1+\cdots+X_n)$ as $n\to\infty$? In which sense?

Exercise 6 Let X be a standard Gaussian random variable. Prove that for any $n\in\mathbb{N}^*$, $\mathbb{E}(X^{2n+1})=0$ and $\mathbb{E}(X^{2n})=\frac{(2n)!}{2^n n!}$. You could for example use an expansion of the characteristic function of X.

Exercise 7. Let $S_n=\sum_{k=1}^n X_1$ where the X_i's are i.i.d. and $\mathbb{P}(X_1=1)=p$, $\mathbb{P}(X_1=0)=1-p$. Prove that for any $\varepsilon>0$, $\mathbb{P}(S_n/n>p+\varepsilon)\leq e^{-\frac{1}{4}n\varepsilon^2}$.

Exercise 8. Give the proof of the following result, seen in class: convergence in probability implies almost sure convergence along a subsequence.