Probability, final training.

Eight exercises of this type will be proposed, you will have 100 minutes. Grade will be scaled so that six exercises perfectly solved will give maximal score 100/100.

Exercise 1. Let X be a real random variable uniform on $[0,1]$.

(i) Calculate $\mathrm{Var}(X)$.
(ii) What is the density of $1/X$?

Exercise 2. Seven people throw their hats in a box and then randomly redistribute among themselves (each person getting one hat, all 7! permutations being equally likely). Let N be the number of people who get their own hat back. Compute the following:

(i) $\mathbb{E}(N)$;
(ii) $\mathbb{P}(N=6)$;
(iii) $\mathbb{P}(N=5)$.

Exercise 3. Let X_1, X_2, X_3 be independent standard die rolls, each outcome from $\{1,\dots,6\}$ being equally likely. Write $Z = X_1 + X_2 + X_3$. Compute

$$\mathbb{P}(X_1 = 6 \mid Z = 16).$$

Exercise 4. Let X be a real Gaussian random variable with mean 0 and variance 7. Calculate

(i) $\mathbb{E}(e^{\lambda X})$ for any $\lambda \in \mathbb{C}$;
(ii) $\mathbb{E}(X^7 - 3X^2 + 12X - 4)$.

Exercise 5. Assume $\mathbb{P}(A) = 3/4$ and $\mathbb{P}(B) = 1/3$. What are possible values of $\mathbb{P}(A \cap B)$?

Exercise 6. Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$. Define $Z_k = \prod_{i=1}^k X_i$. Prove that $(Z_k)_{k\geq 1}$ is a sequence of independent random variables.

Exercise 7. A sequence of random variables $(X_i)_{i\geq 1}$ is said to be completely convergent to X if for any $\varepsilon > 0$, we have $\sum_{i\geq 1} \mathbb{P}(|X_i - X| > \varepsilon) < \infty$. Prove that if the X_i's are independent then complete convergence implies almost sure convergence.

Exercise 8. Let X, Y be independent and assume that for some constant α we have $\mathbb{P}(X + Y = \alpha) = 1$. Prove that both X and Y are both constant random variables.

Exercise 8. Let X be a standard Cauchy random variable. What is the characteristic function of $1/X$?

Exercise 9. Let X have density $\frac{1}{2}e^{-|x|}$. What is the characteristic function of X?

Exercise 10. Let $(X_j)_{j\geq 1}$ be i.i.d. positive with $\log X_j \in \mathrm{L}^4$. Prove that $(\prod_{j=1}^n X_j)^{1/n}$ converges almost surely as $n \to \infty$. Does it converge in distribution?

Exercise 11. Show that

$$\lim_{n\to\infty} e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!}=\frac{1}{2}.$$

Exercise 12. Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with mean 0 and finite variance $\mathbb{E}(X_j^2)=\sigma^2>0$. Let $S_n=X_1+\cdots+X_n$. Prove that

$$\lim_{n\to\infty}\mathbb{E}\left(\frac{|S_n|}{\sqrt{n}}\right)=\sqrt{\frac{2}{\pi}}\sigma.$$

Exercise 13. Let $(X_i)_{i\geq 1}$ be a sequence of independent random variables, with X_i uniform on $[-i,i]$. Let $S_n=X_1+\cdots+X_n$. Prove that $S_n/n^{3/2}$ converges in distribution and describe the limit.

Exercise 14. Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables with expectation 0 and finite variance. Let $S_n=\sum_{\ell=1}^n X_\ell$. Prove that for any $\lambda>0$,

$$\lambda^2\,\mathbb{P}(\max_{1\leq k\leq n}|S_k|\geq\lambda)\leq \mathrm{Var}(S_n).$$

Prove that if $\sum_\ell \mathrm{Var}(X_\ell)<\infty$, then $(S_n)_{n\geq 1}$ converges almost surely.