Probability, homework 1, due September 16.

Exercise 1. Prove whether the following sets are countable or not.

(i) All intervals in $\mathbb{R}$ with rational endpoints.
(ii) All circles in the plane with rational radii and centers on the diagonal $x = y$.
(iii) All sequences of integers whose terms are either 0 or 1.

Exercise 2.

(i) A family has 5 children, consisting of 3 girls and 2 boys. Assuming that all birth orders are equally likely, what is the probability that the 3 eldest children are girls?
(ii) How many ways are there to split 11 people into 3 teams, where one team has 2 people, one has 4 and the other 5?

Exercise 3. Assume you attend a college in which each class meets only once a week. You decide between 30 non-overlapping classes. There are 6 classes to choose from for each day of the week, Monday through Friday. Trusting randomness, you decides to register for 7 randomly selected classes out of the 30, with all choices equally likely. What is the probability that you will have classes every day, Monday through Friday?

Exercise 4. Let X be a random variable with geometric distribution parameterized by $q \in (0, 1)$:

$$\mathbb{P}(X = k) = (1 - q)^k q, \ k = 0, 1, 2 \dots$$

Calculate the variance of X.

Exercise 5. Let $\varnothing \subsetneq A \subsetneq B \subsetneq \Omega$ (these are strict inclusions). What is the σ-algebra generated by $\{A, B\}$?

Exercise 6. Proof of Stirling's formula.

(i) Using a trapezoidal approximation for the area under a curve, prove that

$$\sum_{k=1}^{n} \log k = \left(n + \frac{1}{2}\right) \log n - n + c + \mathrm{o}(1)$$

as $n \to \infty$, where c is a non-explicit constant.

(ii) Considering the the simple random walk S discussed in class, show that $c = \log(\sqrt{2\pi})$ and conclude the proof of Stirling's formula. Hint: write

$$\sum_{k=-\infty}^{\infty} \mathbb{P}(S_n = k) = 1$$

and take n to ∞.