Probability, homework 6, due November 1st.

Exercise 1. Let X have a binomial distribution with parameters (p, n). Prove that X is even with probability

$$\frac{1}{2}\left(1+(1-2p)^n\right).$$

Exercise 2. Let X, Y be independent random variables with positive integers values, with distribution

$$\mathbb{P}(X=i)=\mathbb{P}(Y=i)=\frac{1}{2^i}, i \in \mathbb{N}^*.$$

Find the following proabilitities.

(i) $\mathbb{P}(\max(X,Y) \geq i)$.
(ii) $\mathbb{P}(X=Y)$.
(iii) $\mathbb{P}(X>Y)$.

Exercise 3. Suppose a distribution function F of a random variable X is given by

$$F(x)=\frac{1}{4}\mathbb{1}_{[0,\infty)}(x)+\frac{1}{2}\mathbb{1}_{[1,\infty)}(x)+\frac{1}{4}\mathbb{1}_{[2,\infty)}(x)$$

What is the probability that X belongs to the following sets, $(-1/2, 1/2)$, $(-1/2, 3/2)$, $(2/3, 5/2)$, $(3, \infty)$?

Exercise 4. Let X be uniformly distributed on $[0,1]$ and $\lambda > 0$. Show that $-\lambda^{-1}\log X$ has the same distribution as an exponential random variable with parameter λ.

Exercise 5. Let X be a standard Gaussian random variable. What is the density of $1/X^2$?

Exercise 6. Let X be a positive random variable with density $e^{-x}\mathbb{1}_{x>0}$ (the exponential distribution). What is the density of $1/(1+X)$?

Exercise 7 Let X be a standard Gaussian random variable. Prove that for any $n \in \mathbb{N}^*$, $\mathbb{E}(X^{2n+1})=0$ and $\mathbb{E}(X^{2n})=\frac{(2n)!}{2^n n!}$.

Exercise 8. The goal of this exercise is to prove that any function, continuous on an interval of $\mathbb{R}$, can be approximated by polynomials, arbitrarily close for the L^∞ norm (this is the Bernstein-Weierstrass theorem). Let f be a continuous function on $[0,1]$. The n-th Bernstein polynomial is

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right).$$

(i) Let $S_n(x)=B^{(n,x)}/n$, where $B^{(n,x)}$ is a binomial random variable with parameters n and x: $B^{(n,x)}=\sum_{\ell=1}^{n}X_i$ where the X_i's are independent and $\mathbb{P}(X_i=1)=x$, $\mathbb{P}(X_i=0)=1-x$. Prove that $B_n(x)=\mathbb{E}(f(S_n(x)))$.
(ii) Prove that $\|B_n-f\|_{L^\infty([0,1])}\to 0$ as $n\to\infty$ and conclude.