Exercise 1. Let $X$ have a binomial distribution with parameters $(p, n)$. Prove that $X$ is even with probability
$$\frac{1}{2} (1 + (1 - 2p)^n).$$

Exercise 2. Let $X, Y$ be independent random variables with positive integer values, with distribution
$$P(X = i) = P(Y = i) = \frac{1}{2^i}, i \in \mathbb{N}^*.$$
Find the following probabilities.
(i) $P(\max(X, Y) \geq i)$.
(ii) $P(X = Y)$.
(iii) $P(X > Y)$.

Exercise 3. Suppose a distribution function $F$ of a random variable $X$ is given by
$$F(x) = \frac{1}{4}\mathbb{1}_{[0, \infty)}(x) + \frac{1}{2}\mathbb{1}_{[1, \infty)}(x) + \frac{1}{4}\mathbb{1}_{[2, \infty)}(x).$$
What is the probability that $X$ belongs to the following sets, $(-1/2, 1/2)$, $(-1/2, 3/2)$, $(2/3, 5/2)$, $(3, \infty)$?

Exercise 4. Let $X$ be uniformly distributed on $[0, 1]$ and $\lambda > 0$. Show that $-\lambda^{-1}\log X$ has the same distribution as an exponential random variable with parameter $\lambda$.

Exercise 5. Let $X$ be a standard Gaussian random variable. What is the density of $1/X^2$?

Exercise 6. Let $X$ be a positive random variable with density $e^{-x}\mathbb{1}_{x > 0}$ (the exponential distribution). What is the density of $1/(1 + X)$?

Exercise 7. Let $X$ be a standard Gaussian random variable. Prove that for any $n \in \mathbb{N}^*$, $E(X^{2n+1}) = 0$ and $E(X^{2n}) = \frac{(2n)!}{2^{n+1}n!}$.

Exercise 8. The goal of this exercise is to prove that any function, continuous on an interval of $\mathbb{R}$, can be approximated by polynomials, arbitrarily close for the $L^\infty$ norm (this is the Bernstein-Weierstrass theorem). Let $f$ be a continuous function on $[0, 1]$. The $n$-th Bernstein polynomial is
$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} f\left(\frac{k}{n}\right).$$
(i) Let $S_n(x) = B^{(n,x)}/n$, where $B^{(n,x)}$ is a binomial random variable with parameters $n$ and $x$: $B^{(n,x)} = \sum_{i=1}^{n} X_i$ where the $X_i$’s are independent and $P(X_i = 1) = x$, $P(X_i = 0) = 1 - x$. Prove that $B_n(x) = E(f(S_n(x)))$.
(ii) Prove that $\|B_n - f\|_{L^\infty([0,1])} \rightarrow 0$ as $n \rightarrow \infty$ and conclude.