Probability, homework 1.

Exercise 1.
(i) A family has 5 children, consisting of 3 girls and 2 boys. Assuming that all birth orders are equally likely, what is the probability that the 3 eldest children are girls?
(ii) How many ways are there to split 11 people into 3 teams, where one team has 2 people, one has 4 and the other 5?

Exercise 2. Prove that
\[
\binom{n+m}{r} = \binom{n}{0} \binom{m}{r} + \binom{n}{1} \binom{m}{r-1} + \cdots + \binom{n}{r} \binom{m}{0}.
\]

Exercise 3. The following identity is known as Fermat’s combinatorial identity:
\[
\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1}, \quad n \geq k.
\]
Give a combinatorial argument (no computations are needed) to establish this identity. Hint: Consider the set of numbers 1 through \( n \). How many subsets of size \( k \) have \( i \) as their highest-numbered member?

Exercise 4. Proof of Stirling’s formula. Using a trapezoidal approximation for the area under a curve, prove that
\[
\sum_{k=1}^{n} \log k = \left( n + \frac{1}{2} \right) \log n - n + c + o(1)
\]
as \( n \to \infty \), where \( c \) is a non-explicit constant.