Probability, homework 10, due April 15.

Exercise 1. *Sufficient condition for convergence in distribution.* Assume that the sequence of random variables $(X_n)_{n\geq 1}$, X are such that

$$\mathbb{E}\, f(X_n) \to \mathbb{E}\, f(X)$$

or any smooth and compactly supported unction f. Prove that X_n converges to X in distribution.

Exercise 2. Assume that the sequence of random variables $(X_n)_{n\geq 1}$ satisfies $\mathbb{E}\, X_n \to 1$ and $\mathbb{E}\, X_n^2 \to 1$. Prove that $(X_n)_{n\geq 1}$ converges in distribution. What is the limit?

Exercise 3. *Convergence in L^1 in the strong law of large numbers.*

a) Read online (or in Jacod-Protter) the definition of a uniformly integrable sequence of random variables.

b) Prove that if S_n converges to S almost surely, and $(S_n)_{n\geq 1}$ is uniformly integrable, then S_n converges to S in L^1.

c) Prove that if the X_ℓ's are i.i.d. and in L^1, then $(n^{-1}\sum_{k=1}^n X_k)_{n\geq 1}$ is uniformly integrable.

d) Conclude that the strong law of large numbers in the almost sure sense for random variables in L^1 implies the strong law of large numbers in the L^1 sense.

Exercise 4. Let $(X_n)_{n\geq 1}$ be a sequence of random variables, on the same probability space, with $\mathbb{E}(X_\ell) = \mu$ for any ℓ, and a weak correlation in the following sense: $\mathrm{Cov}(X_k, X_\ell) \leq f(|k-\ell|)$ for all indexes k, ℓ, where the sequence $(f(m))_{m\geq 0}$ converges to 0 as $m \to \infty$. Prove that $(n^{-1}\sum_{k=1}^n X_k)_{n\geq 1}$ converges to μ in L^2.

Exercise 5 Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables, on the same probability space, with law given by $\mathbb{P}(X_1 = (-1)^m m) = 1/(cm^2 \log m)$ for $m \geq 2$ (c is the normalization constant $c = \sum_{m\geq 2} 1/(m^2 \log m)$). Prove that $\mathbb{E}(|X_1|) = \infty$, but there exists a constant $\mu \notin \{\pm\infty\}$ such that $(n^{-1}\sum_{k=1}^n X_k)_{n\geq 1}$ converges to μ in probability. Does it converge almost surely, and in L^p?

Exercise 6. Let the X_ℓ's be independent uniformly bounded real random variables. Let $\mu_\ell = \mathbb{E}(X_\ell)$, and $\sigma_\ell^2 = \mathrm{Var}(X_\ell)$ satisfy $c_1 < \sigma_\ell^2$ for some c_1 which does not depend on ℓ. State and prove a central limit theorem for $\sum_{\ell=1}^n X_\ell$.