Exercise 1.
(i) A family has 5 children, consisting of 3 girls and 2 boys. Assuming that all birth orders are equally likely, what is the probability that the 3 eldest children are girls?
(ii) How many ways are there to split 11 people into 3 teams, where one team has 2 people, one has 4 and the other 5?

Solution.
(i) Let \( \sigma \) be a uniform permutation of \( \{1, 2, 3, 4, 5\} \). We are asked to find
\[
P(\{\sigma(1), \sigma(2), \sigma(3)\} = \{1, 2, 3\}) = P(\{\sigma(4), \sigma(5)\} = \{4, 5\})
\]
\[
= P((\sigma(4), \sigma(5)) = (4, 5)) + P((\sigma(4), \sigma(5)) = (5, 4)).
\]
Each probability above is \( 3! / 5! \) (Obviously \( 3! \) choices for \( (\sigma(1), \sigma(2), \sigma(3)) \)), so the final answer is \( 2 \cdot 3! / 5! = 1 / 10 \).

(ii) If \( S_{11} \) denotes the group of permutations of 11 elements. For \( \sigma, \tau \in \Sigma_{11} \), we define \( \sigma \sim \tau \) if \( \{\sigma(i), 1 \leq i \leq 2\} = \{\tau(i), 1 \leq i \leq 2\}, \{\sigma(i), 3 \leq i \leq 7\} = \{\tau(i), 3 \leq i \leq 7\} \) and \( \{\sigma(i), 8 \leq i \leq 11\} = \{\tau(i), 8 \leq i \leq 11\} \). This is an equivalence relation. We are asked to count the number of equivalence classes. This is
\[
\frac{|S_{11}|}{|S_2| \cdot |S_5| \cdot |S_4|} = \frac{11!}{2!5!4!} = 6930.
\]
Note that you also can directly invoke the multinomial formula.

Exercise 2. Prove that
\[
\binom{n+m}{r} = \sum_{i=0}^{r} \binom{n}{i} \binom{m}{r-i}.
\]

Solution. Let \( A_1 \) and \( A_2 \) be disjoint sets, \( |A_1| = n, |A_2| = m \). Then
\[
\{C : C \subset A_1 \cup A_2, |C| = r\} = \bigsqcup_{k=0}^{r} \{C_1 \sqcup C_2 : C_1 \subset A_1, C_2 \subset A_2, |C_1| = k, |C_2| = r-k\},
\]
where all unions are disjoint. Taking cardinality in the above set equality gives the result.

Exercise 3. The following identity is known as Fermat’s combinatorial identity:
\[
\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1}, \quad n \geq k.
\]
Give a combinatorial argument (no computations are needed) to establish this identity. Hint: Consider the set of numbers 1 through \( n \). How many subsets of size \( k \) have \( i \) as their highest-numbered member?

Solution. We have
\[
\{C : C \subset \mathbb{[1, n]}, |C| = k\} = \bigsqcup_{i=k}^{n} \{D \sqcup \{i\} : D \subset \mathbb{[1, i-1]}, |D| = k-1\}.
\]
where all unions are disjoint. Taking cardinality in the above set equality gives the result.

**Exercise 4.** Proof of Stirling’s formula. Using a trapezoidal approximation for the area under a curve, prove that

\[ \sum_{k=1}^{n} \log k = \left( n + \frac{1}{2} \right) \log n - n + c + o(1) \]

as \( n \to \infty \), where \( c \) is a non-explicit constant.

*Solution.* As \( \log''(x) = 1/x^2 \), the function \( \log \) is concave. As a consequence, for any \( k \geq 1 \) and \( x \in [k, k+1] \),

\[
\log k + (x - k)(\log(k + 1) - \log k) < \log x < \log k + (x - k)\log'(k)
\]

Integrating the above on \([k, k+1]\) gives

\[
\frac{\log k + \log(k + 1)}{2} < \int_{k}^{k+1} (\log x) dx < \log k + \frac{1}{2k},
\]

so that

\[
0 < \log k - \left( \int_{k}^{k+1} (\log x) dx - \frac{1}{2k} \right) < \frac{\log k - \log(k + 1)}{2} + \frac{1}{2k} = O(k^{-2}).
\]

Summation over \( k \) allows to conclude.