Exercise 1. Let \((G_\alpha)_{\alpha \in A}\) be an arbitrary family of \(\sigma\)-algebras defined on an abstract space \(\Omega\). Show that \(\bigcap_{\alpha \in A} G_\alpha\) is also a \(\sigma\)-algebra.

Exercise 2. Let \(\mathcal{A}\) be a \(\sigma\)-algebra. Prove that if, for all \(n \in \mathbb{N}\), \(A_n \in \mathcal{A}\), then \(\limsup_{n \to \infty} A_n\) and \(\liminf_{n \to \infty} A_n\) are in \(\mathcal{A}\) (these limiting events are defined in Jacod-Protter).

Exercise 3. Prove the Bonferroni inequalities: if \(A_i \in \mathcal{A}\) is a sequence of events, then

(i) \(\mathbb{P}(\bigcup_{i=1}^n A_i) \geq \sum_{i=1}^n \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j)\),

(ii) \(\mathbb{P}(\bigcup_{i=1}^n A_i) \leq \sum_{i=1}^n \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k)\).

Exercise 4. Prove whether the following sets are countable or not.

(i) All intervals in \(\mathbb{R}\) with rational endpoints.
(ii) All circles in the plane with rational radii and centers on the diagonal \(x = y\).
(iii) All sequences of integers whose terms are either 0 or 1.

Exercise 5. Let \(\emptyset \subset A \subset B \subset \Omega\) (these are strict inclusions). What is the \(\sigma\)-algebra generated by \(\{A, B\}\)?

Exercise 6. Let \((s_n)_{n \geq 0}\) be a random walk. For \(a \in \mathbb{Z}^*\), let \(T_a = \inf\{n \geq 0 : s_n = a\}\). Prove that \(\mathbb{E}(T_a) = \infty\).