Exercise 1. Let \((G_\alpha)_{\alpha \in A}\) be an arbitrary family of \(\sigma\)-algebras defined on an abstract space \(\Omega\). Show that \(\bigcap_{\alpha \in A} G_\alpha\) is also a \(\sigma\)-algebra.

**Solution.** We need to check that

(i) \(\emptyset \in \bigcap_{\alpha \in A} G_\alpha\),
(ii) if \(A \in \bigcap_{\alpha \in A} G_\alpha\), then \(A^c \in \bigcap_{\alpha \in A} G_\alpha\),
(iii) if \((A_i)_{i \geq 1}\) is a sequence in \(\bigcap_{\alpha \in A} G_\alpha\), then \(\bigcup_{i \geq 1} A_i \in \bigcap_{\alpha \in A} G_\alpha\) and \(\bigcap_{i \geq 1} A_i \in \bigcap_{\alpha \in A} G_\alpha\).

Let us check just the last statement on intersections, all others are proved similarly and left to the reader. If \(A_i \in \bigcap_{\alpha \in A} G_\alpha\), for any \(\alpha\) we have \(A_i \in G_\alpha\). As \(G_\alpha\) is a \(\sigma\)-algebra this implies \(\bigcap_{i \geq 1} A_i \in G_\alpha\). As this is true for any \(\alpha\), we have \(\bigcap_{i \geq 1} A_i \in \bigcap_{\alpha \in A} G_\alpha\).

Exercise 2. Let \(A\) be a \(\sigma\)-algebra. Prove that if, for all \(n \in \mathbb{N}\), \(A_n \in A\), then \(\limsup_{n \to \infty} A_n\) and \(\liminf_{n \to \infty} A_n\) are in \(A\) (these limiting events are defined in Jacod-Protter).

**Solution.** By definition we have

\[\limsup_{n \to \infty} A_n = \bigcap_{n \geq 1} \bigcup_{m \geq n} A_m.\]

For any given \(n\), \(\bigcup_{m \geq n} A_m \in A\) as this is a countable union of elements in \(A\). This \(\bigcap_{n \geq 1} \bigcup_{m \geq n} A_m\) is a countable intersections of elements in \(A\), thus in \(A\). The proof for the lim inf is the same.

Exercise 3. Prove the Bonferroni inequalities: if \(A_i \in A\) is a sequence of events, then

(i) \(P(\bigcup_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j)\),
(ii) \(P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k)\).

**Solution.** For any \(x \in \Omega\) we claim that

\[\mathbf{1}_{x \in \bigcup_{i=1}^{n} A_i} \geq \sum_{i} \mathbf{1}_{x \in A_i} - \sum_{i<j} \mathbf{1}_{x \in A_i \cap A_j} ,\]

(0.1)

Indeed, if \(x\) is in exactly \(m\) sets \(A_i\)'s, for \(m = 0\) the above relation is 0 \(\geq 0\) and if \(m \geq 1\) this is

\[1 \geq m - \binom{m}{2},\]

which is true. Integrating (0.1) w.r.t \(P\) gives the result.

The same type of reasoning holds for (ii), stopping the binomial expansion at third order.

Exercise 4. Prove whether the following sets are countable or not.

(i) All intervals in \(\mathbb{R}\) with rational endpoints.
(ii) All circles in the plane with rational radii and centers on the diagonal \(x = y\).
(iii) All sequences of integers whose terms are either 0 or 1.

**Solution.** (i) This is in bijection with \( \mathbb{Q}^2 \), which is countable.

(ii) Let \( S \) be the set of interest. The map \( \phi : S \rightarrow \mathbb{R}, C(x,r) \mapsto x \) (\( x \) is the center, \( r \) the radius) is surjective and \( \mathbb{R} \) is uncountable, so \( S \) is uncountable.

(iii) The following is Cantor’s diagonal argument. Assume the set of interest, \( S \), is countable. Then there exists a bijective map \( \phi : \mathbb{N} \rightarrow S \). We denote \( \phi(k) = (e^{(1)}_k, e^{(2)}_k, \ldots) \) the corresponding sequence of 0’s or 1’s. Then there is no \( i \) such that \( \phi(i) = (1 - e^{(1)}_1, 1 - e^{(2)}_2, \ldots) \) because the \( i \)th coordinate would give \( e^{(i)}_i = 1 - e^{(i)}_i \). This impossibility to find \( i \) such that \( \phi(i) = (1 - e^{(1)}_1, 1 - e^{(2)}_2, \ldots) \) contradicts the claimed bijectivity. Hence \( S \) is uncountable.

**Exercise 5.** Let \( \emptyset \subset A \subset B \subset \Omega \) (these are strict inclusions). What is the \( \sigma \)-algebra generated by \( \{A, B\} \)?

**Solution.** It is clear that \( \{\emptyset, A, B, A^c, B^c, B - A, (B - A)^c, \Omega\} \subset \sigma(\{A, B\}) \). On the other hands, stability of this left hand side by complement, union, intersection can be checked by hand.

**Exercise 6.** Let \( (s_n)_{n \geq 0} \) be a random walk. For \( a \in \mathbb{Z}^* \), let \( T_a = \inf\{n \geq 0 : s_n = a\} \). Prove that \( \mathbb{E}(T_a) = \infty \).

**Solution.** By symmetry we can assume \( a > 0 \). Let \( m_n = \max_{1 \leq k \leq n} s_k \). Then

\[
\mathbb{E}(T_a) = \sum_{n \geq 1} n \mathbb{P}(T_a = n) = \sum_{n \geq 1} \mathbb{P}(T_a \geq n) = \sum_{n \geq 1} \mathbb{P}(m_n \leq a).
\]

By the reflection principle we know that \( \mathbb{P}(m_n \leq a) = \mathbb{P}(|s_n| \leq a) = \mathbb{P}(s_n = a) \geq \mathbb{P}(s_n = a - 1) \sim_{n \rightarrow \infty} \frac{c}{\sqrt{n}} \) where we used Stirling’s asymptotics. This concludes the proof because \( \sum_{n \geq 1} n^{-1/2} \) diverges.