Probability, homework 7, due March 25.

Exercise 1. Let X have a binomial distribution with parameters (p, n). Prove that X is even with probability

$$\frac{1}{2}\left(1+(1-2p)^n\right).$$

Exercise 2

(i) Let X, Y be two independent and identically distributed real random variables. What is $\mathbb{P}(X = Y)$?

(ii) Let $(X_n)_{n\geq 1}$ be a sequence of real, independent and identically distributed random variables, with distribution function F. Show that almost surely we have

$$\max(X_1, \dots, X_n) \to \sup\{x \in \mathbb{R} \mid F(x) < 1\}.$$

Exercise 3. Let X have distribution function $F(x) = e^{-e^{-x}}$. Justify that such a probability measure on $\mathbb{R}$ exists. Let $Y = F(X)$. Calculate $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$.

Exercise 4. Let X be a real Gaussian random variable with mean 0 and variance 7. Calculate

(i) $\mathbb{E}(e^{\lambda X})$ for any $\lambda \in \mathbb{C}$;

(ii) $\mathbb{E}(X^7 - 3X^2 + 12X - 4)$.

Exercise 5 Let X be a standard Gaussian random variable. Prove that for any $n \in \mathbb{N}^*$, $\mathbb{E}(X^{2n+1}) = 0$ and $\mathbb{E}(X^{2n}) = \frac{(2n)!}{2^n n!}$. You could for example use an expansion of the characteristic function of X.

Exercise 6. Let X be a random variable with density $f_X(x) = (1 - |x|)\mathbb{1}_{(-1,1)}(x)$. Show that its characteristic function is

$$\phi_X(u) = \frac{2(1-\cos u)}{u^2}.$$

Exercise 7 Let X be a Poisson random variable with parameter λ. What is its characteristic function?

Exercise 8 Read about the Cauchy residue formula in complex analysis. Applying it to a contour with semicircular shape, prove that for $\mu(\mathrm{d}x) = \frac{1}{\pi(1+x^2)}\mathrm{d}x$ we have $\hat{\mu}(u) = e^{-|u|}$.

Exercise 9

(1) Prove that $\hat{\mu}$ is real-valued if and only if μ is symmetric, i.e. $\mu(A) = \mu(-A)$ for any Borel set A

(2) If X and Y are i.i.d., prove that $X - Y$ has a symmetric distribution.

Exercise 10 Let X, Y be i.i.d., with characteristic functions denoted φ_X, φ_Y, and suppose $\mathbb{E}(X) = 0$, $\mathbb{E}(X^2) = 1$. Assume also that $X+Y$ and $X-Y$ are independent.

(1) Prove that

$$\varphi_X(2u) = (\varphi_X(u))^3 \varphi_X(-u)$$

(2) Prove that X is a standard Gaussian random variable.

Exercise 11 Assume that (X, Y) has joint density

$$ce^{-(1+x^2)(1+y^2)},$$

where c is properly chosen. Prove that X and Y are Gaussian random variables, but that (X, Y) is not a Gaussian vector.