Exercise 1. Prove whether the following sets are countable or not.

(i) All intervals in $\mathbb{R}$ with rational endpoints.
(ii) All circles in the plane with rational radii and centers on the diagonal $x = y$.
(iii) All sequences of integers whose terms are either 0 or 1.

Exercise 2. Let $(\mathcal{G}_\alpha)_{\alpha \in A}$ be an arbitrary family of $\sigma$-fields defined on an abstract space $\Omega$, with $A$ possibly uncountable. Show that $\bigcap_{\alpha \in A} \mathcal{G}_\alpha$ is also a $\sigma$-field.

Exercise 3. Let $\emptyset \subseteq A \subseteq B \subseteq \Omega$ (these are strict inclusions). What is the $\sigma$-field generated by $\{A, B\}$?

Exercise 4. Let $\mathcal{F}$, $\mathcal{G}$ be $\sigma$-fields for the same $\Omega$. Is $\mathcal{F} \cup \mathcal{G}$ a $\sigma$-field?

Exercise 5. For $\Omega = \mathbb{N}$ and $n \geq 0$, let $\mathcal{F}_n = \sigma(\{\{0\}, \ldots, \{n\}\})$. Show that $(\mathcal{F}_n)_{n \geq 0}$ is a non-decreasing sequence but that $\bigcup_{n \geq 0} \mathcal{F}_n$ is not a $\sigma$-field.

Exercise 6. Let $\Omega$ be an infinite set (countable or not). Let $\mathcal{A}$ be the set of subsets of $\Omega$ that are either finite or with finite complement in $\Omega$. Prove that $\mathcal{A}$ is a field but not a $\sigma$-field.

Exercise 7. A monotone class is a collection $\mathcal{M}$ of sets closed under both monotone increasing and monotone decreasing (i.e. if $A_i \in \mathcal{M}$ and either $A_i \uparrow A$ or $A_i \downarrow A$ then $A \in \mathcal{M}$).

Prove that if $\mathcal{A} \subseteq \mathcal{M}$ with $\mathcal{A}$ a field and $\mathcal{M}$ a monotone class, then $\sigma(\mathcal{A}) \subseteq \mathcal{M}$.

Exercise 8. Let $\mathbb{P}$ be a probability measure on $\Omega$, endowed with a $\sigma$-field $\mathcal{A}$.

(i) What is the meaning of the following events, where all $A_n$’s are elements of $\mathcal{A}$?

$$\liminf_{n \to \infty} A_n = \bigcup_{n \geq 1} \bigcap_{k \geq n} A_k, \quad \limsup_{n \to \infty} A_n = \bigcap_{n \geq 1} \bigcup_{k \geq n} A_k.$$  

(ii) Prove that $\limsup_{n \to \infty} A_n$ and $\liminf_{n \to \infty} A_n$ are in $\mathcal{A}$.

(iii) In the special case $\Omega = \mathbb{R}$, for any $p \geq 1$, let

$$A_{2p} = \left[ -1, 2 + \frac{1}{2p} \right], \quad A_{2p+1} = \left( -2 - \frac{1}{2p+1}, 1 \right].$$

What are $\liminf_{n \to \infty} A_n$ and $\limsup_{n \to \infty} A_n$?

(iv) Prove that the following always holds:

$$\mathbb{P} \left( \liminf_{n \to \infty} A_n \right) \leq \liminf_{n \to \infty} \mathbb{P}(A_n), \quad \mathbb{P} \left( \limsup_{n \to \infty} A_n \right) \geq \limsup_{n \to \infty} \mathbb{P}(A_n).$$
Exercise 9. The symmetric difference of two events $A$ and $B$, denoted $A \triangle B$, is the event that precisely one of them occurs: $A \triangle B = (A \cup B) \setminus (A \cap B)$.

(i) Write a formula for $A \triangle B$ that only involves the operations of union, intersection and complement, but no set difference.

(ii) Define $d(A, B) = P(A \triangle B)$. Show that for any three events $A$, $B$, $C$,

$$d(A, B) + d(B, C) - d(A, C) = 2 \left( P(A \cap B^c \cap C) + P(A^c \cap B \cap C^c) \right).$$

(iii) Assume $A \subset B \subset C$. Prove that $d(A, C) = d(A, B) + d(B, C)$.

Exercise 10. Prove the Bonferroni inequalities: if $A_i \in \mathcal{A}$ is a sequence of events, then

(i) $P \left( \bigcup_{i=1}^n A_i \right) \geq \sum_{i=1}^n P(A_i) - \sum_{i<j} P(A_i \cap A_j)$,

(ii) $P \left( \bigcup_{i=1}^n A_i \right) \leq \sum_{i=1}^n P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$. 