Exercise 1. Let $\mathcal{A}$ be a $\sigma$-algebra, $\mathbb{P}$ a probability measure and $(A_n)_{n \geq 1}$ a sequence of events in $\mathcal{A}$ which converges to $A$. Prove that

(i) $A \in \mathcal{A}$;
(ii) $\lim_{n \to \infty} \mathbb{P}(A_n) = \mathbb{P}(A)$.

Solution. As a first step, note that $A_n \to A$ can be rephrased as

$$A = \limsup_{n \to \infty} A_n = \liminf_{n \to \infty} A_n,$$

where we remind that $\limsup A_n = \cap_{n \geq 1} \cup_{m \geq n} A_m$ and $\liminf A_n = \cup_{n \geq 1} \cap_{m \geq n} A_m$. In particular this implies that $A \in \mathcal{A}$.

We therefore known that $(B_n)_n := (\cup_{m \geq n} A_m)_n$ decreases to $A$ and $(C_n)_n := (\cap_{m \geq n} A_m)_n$ increases to $A$, so $\mathbb{P}(B_n) \to \mathbb{P}(A)$ and $\mathbb{P}(C_n) \to \mathbb{P}(A)$. As $B_n \subset A_n \subset C_n$, this implies $\mathbb{P}(A_n) \to \mathbb{P}(A)$.

Exercise 2. Suppose a distribution function $F$ is given by

$$F(x) = \frac{1}{4} \mathbb{1}_{[0, \infty)}(x) + \frac{1}{2} \mathbb{1}_{[1, \infty)}(x) + \frac{1}{4} \mathbb{1}_{[2, \infty)}(x).$$

What is the probability of the following events, $(-1/2, 1/2)$, $(-1/2, 3/2)$, $(2/3, 5/2)$, $(3, \infty)$?

Solution: We have

$$\mathbb{P}((-1/2, 1/2)) = F(1/2) - F(-1/2) = 1/4 - 0 = 1/4,$$
$$\mathbb{P}((-1/2, 3/2)) = F(3/2) - F(-1/2) = 3/4 - 0 = 3/4,$$
$$\mathbb{P}((2/3, 5/2)) = F(5/2) - F(2/3) = 1 - 1/4 = 3/4,$$
$$\mathbb{P}((3, \infty)) = F(\infty) - F(3) = 1 - 1 = 0.$$

Exercise 3. Let $\mu$ be the Lebesgue measure on $\mathbb{R}$. Build a sequence of functions $(f_n)_{n \geq 0}$, $0 \leq f_n \leq 1$, such that $\int f_n \, d\mu \to 0$ but for any $x \in \mathbb{R}$, $(f_n(x))_{n \geq 0}$ does not converge.

Solution: Define $f_n(x) = \mathbb{1}_{x \in [f(n), g(n)]}$ where, for $n \in [2^p, 2^{p+1})$, $f(n) = (n - 2^p - 2^{p-1})/p$ and $g(n) = f(n) + 1/p$. Clearly $\int f_n \, d\mu = 1/p \to 0$ and for any $x$ we have $f_n(x) = 1$, and $0$ i.o.

Exercise 4. Let $X$ be a random variable in $L^1(\Omega, \mathcal{A}, \mathbb{P})$. Let $(A_n)_{n \geq 0}$ be a sequence of events in $\mathcal{A}$ such that $\mathbb{P}(A_n) \to 0$. Prove that $\mathbb{E}(X \mathbb{1}_{A_n}) \to 0$.

Solution: For any $C > 0$ we have

$$|\mathbb{E}(X \mathbb{1}_{A_n})| \leq \mathbb{E}(|X| \mathbb{1}_{|X| > C}) + C \mathbb{P}(A_n).$$

Let $\varepsilon > 0$. By monotone convergence, there exists $C > 0$ such that $\mathbb{E}(|X| \mathbb{1}_{|X| > C}) < \varepsilon$. For this $C$, there exists a $n_0$ such that for any $n > n_0$ we have $\mathbb{P}(A_n) < \varepsilon/C$. Thus we have proved that for $n > n_0$, $|\mathbb{E}(X \mathbb{1}_{A_n})| < 2\varepsilon$, which concludes the proof.

Exercise 5. Let $(d_n)_{n \geq 0}$ be a sequence in $(0, 1)$, and $K_0 = [0, 1]$. We define iteratively $(K_n)_{n \geq 0}$ in the following way. From $K_n$, which is the union of closed disjoint intervals, we define $K_{n+1}$ by removing from each interval of $K_n$ an open interval, centered at the middle of the previous one, with length $d_n$ times the length of the previous one. Let $K = \cap_{n \geq 0} K_n$ ($K$ is called a Cantor set).
(a) Prove that $K$ is an uncountable compact set, with empty interior, and whose points are all accumulation points.

(b) What is the Lebesgue measure of $K$?

Solution. (i) Each $K_n$ being closed, so is $K$. Moreover $K \subset [0, 1]$, so it is compact.

To prove that $K$ is uncountable, consider the following bijection $\varphi : K \to \{0, 1\}^\mathbb{N}$:

If $x \in K$, then $x$ is either in the left or right interval from $K_1$, and define $\varphi(x)_0 = 0$ if $x$ is in the left interval, 1 otherwise. Iterations on $K_2$, etc defines $\varphi(x)$, and $\varphi$ is easily shown to be a bijection.

To prove that $K$ has empty interior, assume $x, y \in I$ for some interval $I \subset K$.
Then for any $n$ we have $x, y$ in the same interval from $K_n$, i.e. $\varphi(x)_n = \varphi(y)_n$. As $\varphi$ is a bijection this imposes $x = y$, so I needs to be a point, i.e. $K$ has empty interior.

Finally, for any $x \in K$, the set $\{y \in K : \forall k \leq n, \varphi(y)_k = \varphi(x)_k\}$ is infinite and its points are at distance at most $2^{-(n+1)}$ from $x$, so $x$ is an accumulation point.

(ii) An easy induction shows that the Lebesgue measure of $K_n$ is $(1-d_0)\ldots(1-d_{n-1})$. So

$$\text{Leb}(K) = \lim_{n \to \infty} (1-d_0)\ldots(1-d_{n-1}).$$

This is 0 if the series of $d_n$'s diverges, a number in (0, 1) otherwise (for this analysis, take the logarithm and Taylor-expand).

Exercise 6. Let $X$ be a nonnegative random variable. Prove that $\mathbb{E}(X) < +\infty$ if and only if $\sum_{n \in \mathbb{N}} \mathbb{P}(X \geq n) < \infty$.

Solution. By monotone convergence we have $\mathbb{E}(X) = \sum_{n \geq 0} \mathbb{E}(X 1_{X \in [n,n+1]})$, so that

$$-1+\sum_{n \geq 0} (n+1)\mathbb{P}(X \in [n,n+1]) = \sum_{n \geq 0} n\mathbb{P}(X \in [n,n+1]) \leq \mathbb{E}(X) \leq \sum_{n \geq 0} (n+1)\mathbb{P}(X \in [n,n+1]).$$

The result follows by noting that $\sum_{n \geq 0} (n+1)\mathbb{P}(X \in [n,n+1]) = \sum_{n \in \mathbb{N}} \mathbb{P}(X \geq n)$.

Exercise 7. Convergence in measure. Let $(\Omega, \mathcal{A}, \mu)$ be a probability space, and $(f_n)_{n \geq 1}, f : \Omega \to \mathbb{R}$ measurable (for the Borel $\sigma$-field on $\mathbb{R}$). We say that $(f_n)_{n \geq 1}$ converges in measure to $f$ if for any $\varepsilon > 0$ we have

$$\mu(|f_n - f| > \varepsilon) \to 0.$$

(i) Show that $\int |f - f_n| \, d\mu \to 0$ implies that $f_n$ converges to $f$ in measure. Is the reciprocal true?

(ii) Show that if $f_n \to f$ $\mu$-almost surely, then $f_n \to f$ in measure. Is the reciprocal true?

(iii) Show that if $f_n \to f$ in measure, there exists a subsequence of $(f_n)_{n \geq 1}$ which converges $\mu$-almost surely.

(iv) (A stronger dominated convergence theorem) We assume that $f_n \to f$ in measure and $|f_n| \leq g$ for some integrable $g : \Omega \to \mathbb{R}$, for any $n \geq 1$.

(a) Show that $|f| \leq g$ $\mu$-a.s.

(b) Deduce that $\int |f_n - f| \, d\mu \to 0$.

Solution.
(i) For any \( \varepsilon > 0 \) we have \( \mathbf{1}_{|f - f_n| > \varepsilon} \leq \frac{|f - f_n|}{\varepsilon} \), so that
\[
\mu(|f - f_n| > \varepsilon) \leq \frac{1}{\varepsilon} \int_\Omega |f_n - f| \, d\mu \to 0.
\]

The reciprocal is wrong, as shown by the example \((f_n)_{n \geq 1}\) defined on \([0, 1], \mathcal{B}, \text{Leb}\) by \(f_n = n \mathbf{1}_{[0, 1/n]}\).

(ii) For any \( \varepsilon > 0 \) we have
\[
\cap_{n \geq 1} \cup_{m \geq n} \{|f_n - f| > \varepsilon\} \subset \{f_n \to f\}
\]
so that \(\mu(\cap_{n \geq 1} \cup_{m \geq n} \{|f_n - f| > \varepsilon\}) = 0\). The sequence \((\cup_{m \geq n} \{|f_m - f| > \varepsilon\})_{n \geq 1}\) decreases, so this implies that
\[
\lim_{n \to \infty} \mu(\cup_{m \geq n} \{|f_m - f| > \varepsilon\}) = 0,
\]
and in particular \(\lim_{n \to \infty} \mu(\{|f_n - f| > \varepsilon\}) = 0\).

The reciprocal is wrong, as shown by the example \((f_{n,k})_{n \geq 1, 1 \leq k \leq n}\) defined on \((0, 1], \mathcal{B}, \text{Leb}\) by \(f_{n,k} = \mathbf{1}_{(k-1/n, k/n]}\).

(iii) From the hypothesis, for any \( k \geq 1 \) there exists an index \( n_k \) such that \(\mu(|f_{n_k} - f| > 1/k) \leq 2^{-k}\). Summability in \( k \) easily implies
\[
\mu(\cap_{m \geq 1} \cup_{k \geq m} \{|f_{n_k} - f| > 1/k\}) = 0.
\]

As \( f_{n_k} \to f \) is included in the above set, this concludes the proof.

(iv) (a) For any \( \varepsilon > 0 \) we have
\[
\mu(\{|f| > g + \varepsilon\}) \leq \mu(\{|f| > |f_n| + \varepsilon\}) + \mu(\{|f - f_n| > \varepsilon\})
\]
so that \(\mu(\{|f| > g + \varepsilon\}) = 0\). Therefore \(\mu\)-a.s. for any \( n \geq 1 \) we have \(|f| \leq g + 1/n\), so \(|f| \leq g\).

(b) We have
\[
\int_\Omega |f_n - f| \, d\mu = \int_{|f_n - f| < \varepsilon} |f_n - f| \, d\mu + \int_{|f_n - f| \geq \varepsilon} |f_n - f| \, d\mu \leq \varepsilon + 2 \int_{|f_n - f| \geq \varepsilon} |g| \, d\mu.
\]
As \( g \) is integrable, \(\int_{|f_n - f| \geq \varepsilon} |g| \, d\mu \to 0 \) as \( n \to \infty \) (e.g. by dominated convergence or uniform continuity of the integral), and the conclusion follows as \( \varepsilon \) is arbitrary.

**Exercise 8.** Consider a probability space \((\Omega, \mathcal{A}, \mu)\) and \((A_n)_{n \in \mathbb{N}}\) a sequence in \(\mathcal{A}\). Let \( f : \Omega \to \mathbb{R} \) be measurable (for the Borel \( \sigma \)-field on \( \mathbb{R} \)) such that \(\int_\Omega |1_{A_n} - f| \, d\mu \to 0\) as \( n \to \infty \). Prove that there exists \( A \in \mathcal{A} \) such that \( f = 1_A \mu\)-a.s., i.e. \(\mu(f = 1_A) = 1\).

**Solution.** We first prove that \(|f| < 2 \mu\)-a.s.: \(\{|f| > 2\} \subset \{|f - 1_{A_n}| > 1\}\), so that
\[
\mu(|f| > 2) \leq \mu(|f - 1_{A_n}| > 1) \leq \int_\Omega |1_{A_n} - f| \, d\mu \to 0,
\]
where for the second inequality we used \(\mu(X > 1) \leq \mathbb{E}(X)\) for any positive random variable. This proves \(|f| < 2 \mu\)-a.s.
We now prove that \( f = f^2 \mu\text{-a.s.} \), so that the expected result follows from the choice \( A = \{ f = 1 \} \). We have

\[
\int_\Omega |f - f^2|d\mu \leq \int_\Omega |f - 1_A_n|d\mu + \int_\Omega |f^2 - 1_A_n|d\mu \\
= \int_\Omega |f - 1_A_n|d\mu + \int_\Omega |f - 1_A_n| \cdot |f + 1_A_n|d\mu \leq 4 \int_\Omega 1_{A_n} - f|d\mu \to 0.
\]

where we used that \( \mu\text{-a.s. } |f + 1_A_n| \leq |f| + 1 \leq 3 \). This concludes the proof.