# Probability, homework 3 due September 28.

**Exercise 1**. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Prove that if $A \cap B = \emptyset$ and $A, B$ are independent, then $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.

**Exercise 2**. Let $X$ be a nonnegative random variable with null expectation. Prove that it is 0 almost surely.

**Exercise 3**. Calculate $\mathbb{E}(X)$ for the following probability measures $\mathbb{P}^X$.

(i) $\mathbb{P}^X$ has Gaussian density $\frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$, for some $\sigma > 0$ and $\mu \in \mathbb{R}$;
(ii) $\mathbb{P}^X$ has exponential dentity $\lambda e^{-\lambda x} \mathbb{1}_{x \geq 0}$ for some $\lambda > 0$;
(iii) $\mathbb{P}^X = p\delta_a + q\delta_b$ where $p + q = 1$, $p, q \geq 0$ and $a, b \in \mathbb{R}$;
(iv) $\mathbb{P}^X$ is the Poisson distribution: $\mathbb{P}^X(\{n\}) = e^{-\lambda} \frac{\lambda^n}{n!}$ for any integer $n \geq 0$, for some $\lambda > 0$.

**Exercise 4**. Let $X$ be a standard Gaussian random variable. What is the density of $1/X^2$?

**Exercise 5**. Let $X$ be uniformly distributed on $[0, 1]$ and $\lambda > 0$. Show that $-\lambda^{-1} \log X$ has the same distribution as an exponential random variable with parameter $\lambda$.

**Exercise 6**. A samouraï wants to create a triangle with a (rigid) spaghetti. With his saber, he cuts this spaghetti on two places, chosen uniformly and independently along this traditional pasta. What is the probability that he can create a triangle with sides these three pieces of spaghetti?

**Exercise 7**. Assume that $X_1, X_2, \dots$ are independent random variables uniformly distributed on $[0, 1]$. Let $Y^{(n)} = n \inf\{X_i, 1 \leq i \leq n\}$. Prove that it converges weakly to an exponential random variable, i.e. for any continuous bounded function $f : \mathbb{R}^+ \to \mathbb{R}$,

$$\mathbb{E}\left(f(Y^{(n)})\right) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^+} f(u)e^{-u} \mathrm{d}u.$$

**Exercise 8**. Let $n$ and $m$ be random numbers chosen independently and uniformly on $[\![1, N]\!]$. What are $\Omega, \mathcal{A}$ and $\mathbb{P}$ (which all implicitly depend on $N$) ? Prove that $\mathbb{P}(n \wedge m = 1) \underset{N\to\infty}{\longrightarrow} \zeta(2)^{-1}$ where $\zeta(2) = \prod_{p\in\mathcal{P}}(1 - p^{-2})^{-1} = \sum_{n\geq 1} n^{-2} = \frac{\pi^2}{6}$ (you don't have to prove these equalities). Here $\mathcal{P}$ is the set of prime numbers and $n \wedge m = 1$ means that their greatest common divisor is 1.

**Exercise 9**. Let $\epsilon > 0$ and $X$ be uniformly distributed on $[0, 1]$. Prove that, almost surely (i.e. the following event has probability 1), there exists only a finite number of rationals $\frac{p}{q}$, with $p \wedge q = 1$, such that

$$\left| X - \frac{p}{q} \right| < \frac{1}{q^{2+\epsilon}}.$$

**Exercise 10**. You toss a coin repeatedly and independently. The probability to get a head is $p$, a tail is $1-p$. Let $A_k$ be the following event: $k$ or more consecutive heads occur amongst the tosses numbered $2^k, \dots, 2^{k+1} - 1$. Prove that $\mathbb{P}(A_k \text{ i.o.}) = 1$ if $p \geq 1/2$, 0 otherwise.

Here, i.o. stands for "infinitely often", and $A_k$ i.o. is the event $\cap_{n\geq 1} \cup_{m\geq n} A_m$.