Exercise 1. Prove that if a sequence of real random variables \((X_n)\) converge in distribution to \(X\), and \((Y_n)\) converges in distribution to a constant \(c\), then \(X_n + Y_n\) converges in distribution to \(X + c\).

Exercise 2. Assume that \((X, Y)\) has joint density
\[ e^{-\frac{(1+X^2)(1+Y^2)}{2}}, \]
where \(c\) is properly chosen. Are \(X\) and \(Y\) Gaussian random variables? Is \((X, Y)\) a Gaussian vector?

Exercise 3. Let \((X_i)_{i \geq 1}\) be a sequence of i.i.d. random variables with mean 0 and finite variance \(E(X_i^2) = \sigma^2 > 0\). Let \(S_n = X_1 + \cdots + X_n\). Prove that
\[ \lim_{n \to \infty} E \left( \frac{|S_n|}{\sqrt{n}} \right) = \sqrt{\frac{2}{\pi}} \sigma. \]

Exercise 4. Let the \(X_\ell\)'s be independent uniformly bounded real random variables. Let \(\mu_\ell = E(X_\ell)\), and \(\sigma_\ell^2 = \text{Var}(X_\ell)\) satisfy \(c_1 < \sigma_\ell^2\) for some \(c_1\) which does not depend on \(\ell\). State and prove a central limit theorem for \(\sum_{\ell=1}^{n} X_\ell\).

Exercise 5. Let \((X_i)_{i \geq 1}\) be a sequence of independent random variables, with \(X_i\) uniform on \([-i, i]\). Let \(S_n = X_1 + \cdots + X_n\). Prove that \(S_n/n^{3/2}\) converges in distribution and describe the limit.

Long problem. The goal of this problem is to prove the iterated logarithm law, first for Gaussian random variables. In other words, for \(X_1, X_2\ldots\) i.i.d. standard Gaussian random variables, denoting \(S_n = X_1 + \cdots + X_n\), we have
\[ P \left( \limsup_{n \to \infty} \frac{S_n}{\sqrt{2n \log \log n}} = 1 \right) = 1 \quad (0.1) \]

(1) Prove that
\[ P(X_1 > \lambda) \sim_{\lambda \to \infty} \frac{1}{\lambda \sqrt{2\pi}} e^{-\frac{\lambda^2}{2}}. \]

In the following questions we denote \(f(n) = \sqrt{2n \log \log n}, \lambda > 1, c, c_0 > 0, A_k = \{S_{\lfloor \lambda^k \rfloor} \geq cf(\lambda^k)\}, C_k = \{S_{\lfloor \lambda^{k+1} \rfloor} - S_{\lfloor \lambda^k \rfloor} \geq cf(\lambda^{k+1} - \lambda^k)\}\) and \(D_k = \{\sup_{n \in [\lambda^k, \lambda^{k+1}]} \frac{S_n - S_{\lfloor \lambda^k \rfloor}}{f(\lambda^k)} \geq \alpha\}\).

(2) Prove that for any \(c > 1\) we have \(\sum_{k \geq 1} P(A_k) < \infty\) and
\[ \limsup_{k \to \infty} \frac{S_{\lfloor \lambda^k \rfloor}}{f(\lambda^k)} \leq 1 \text{ a.s.} \]

(3) Prove that for any \(c < 1\) we have \(\sum_{k \geq 1} P(C_k) = \infty\) and
\[ P(C_k \text{ i.o.}) = 1. \]

(4) Let \(\varepsilon > 0\) and choose \(c = 1 - \varepsilon/10\). Prove that almost surely the following inequality holds for infinitely many \(k\):
\[ \frac{S_{\lfloor \lambda^{k+1} \rfloor}}{f(\lambda^{k+1})} \geq c \frac{f(\lambda^{k+1} - \lambda^k)}{f(\lambda^{k+1})} - (1 + \varepsilon) \frac{f(\lambda^k)}{f(\lambda^{k+1})}. \]
(5) By choosing a large enough $\lambda$ in the previous inequality, prove that almost surely
\[ \limsup_{n \to \infty} \frac{S_n}{f(n)} \geq 1. \]

(6) Prove that for any $n \in \left[ \lambda^k, \lambda^{k+1} \right]$ we have
\[ \frac{S_n}{f(n)} \leq \frac{S_{\lceil \lambda^k \rceil}}{f(\lceil \lambda^k \rceil)} + \frac{S_n - S_{\lceil \lambda^k \rceil}}{f(\lceil \lambda^k \rceil)}. \]

(7) Prove that
\[ \mathbb{P}(D_k) \sim 2^k \left( X_1 \geq \frac{\alpha f(\lambda^k)}{\sqrt{\lambda^k+1 - \lambda^k}} \right) \sim \frac{c}{\sqrt{\log \lambda}} \left( \frac{1}{k} \right)^{\frac{\alpha^2}{\lambda-1}}. \]

(8) Prove that for $\alpha^2 > \lambda - 1$, almost surely
\[ \limsup_{n \to \infty} \frac{S_n}{f(n)} \leq \limsup_{n \to \infty} \frac{S_{\lceil \lambda^k \rceil}}{f(\lambda^k)} + \alpha. \]

(9) By choosing appropriate $\lambda$ and $\alpha$, prove that almost surely
\[ \limsup_{n \to \infty} \frac{S_n}{f(n)} \leq 1. \]

(10) State a result similar to (0.1) for i.i.d. uniformly bounded random variables. Which steps in the above proof need to be modified to prove this universality result? How?