Probability, homework 7, due November 2.

Exercise 1. Let X and Y be independent Gaussian random variables with null expectation and variance 1. Show that $\frac{X+Y}{\sqrt{2}}$ and $\frac{X-Y}{\sqrt{2}}$ are also independent $\mathcal{N}(0,1)$.

Exercise 2. Let $S_n = \sum_{k=1}^n X_k$ where the X_k's are i.i.d. and $\mathbb{P}(X_1 = 1) = p$, $\mathbb{P}(X_1 = 0) = 1 - p$. Prove that for any $\varepsilon > 0$, $\mathbb{P}(S_n/n > p + \varepsilon) \leq e^{-\frac{1}{4}n\varepsilon^2}$.

Exercise 3. Let $(X_n)_{n\geq 1}$ be real, i.i.d. random variables and N a random variable independent of $(X_n)_{n\geq 1}$ with values in $\mathbb{N}$. Prove that for any measurable $f : \mathbb{R} \to \mathbb{R}_+$ we have

$$\mathbb{E}\left[\sum_{i=1}^{N} f(X_i)\right] = \mathbb{E}[N] \cdot \mathbb{E}[f(X_1)].$$

Exercise 4. Let $(X_n)_{n\geq 1}$ be i.i.d. Bernoulli random variables with parameter $p \in (0,1)$, i.e. $\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = 0) = p$. Let N be a Poisson random variable with parameter $\lambda > 0$, i.e. for any $k \geq 0$ we have $\mathbb{P}(N = k) = e^{-\lambda}\frac{\lambda^k}{k!}$. Assume N is independent from $(X_n)_{n\geq 1}$.

Let $P = \sum_{i=1}^N X_i$, $F = N - P$.

a) What is the joint distribution of (P, N)?

b) Prove that P and F are independent.

Exercise 5. *The number of buses stopping till time t*. Let $(X_n)_{n\geq 1}$ be i.i.d, random variables on $(\Omega, \mathcal{A}, \mathbb{P})$, X_1 being an exponential random variable with parameter 1. Define $T_0 = 0$, $T_n = X_1 + \cdots + X_n$, and for any $t > 0$,

$$N_t = \max\{n \geq 0 \mid T_n \leq t\}$$

a) For any $n \geq 1$, calculate the joint distribution of $(T_1, \dots, T_n)$.

b) Deduce the distribution of N_t, for arbitrary t.

Exercise 6. *The problem of the collector*. Let $(X_k)_{k\geq 1}$ be a sequence of independent random variables uniformly distributed on $\{1, \dots, n\}$. Let $\tau_n = \inf\{m \geq 1 : \{X_1, \dots, X_m\} = \{1, \dots, n\}\}$ be the first time for which all values have been observed.

a) Let $\tau_n^{(k)} = \inf\{m \geq 1 : |\{X_1, \dots, X_m\}| = k\}$. Prove that the random variables $(\tau_n^{(k)} - \tau_n^{(k-1)})_{2\leq k\leq n}$ are independent and calculate their respective distributions.

b) Deduce that $\frac{\tau_n}{n \log n} \to 1$ in probability as $n \to \infty$, i.e. for any $\varepsilon > 0$,

$$\mathbb{P}\left(\left|\frac{\tau_n}{n \log n} - 1\right| > \varepsilon\right) \to 0.$$

Exercise 7. Let $(X_n)_{n\geq 0}$ be real, independent, random variables on $(\Omega, \mathcal{A}, \mathbb{P})$.

a) Prove that the radius of convergence R of the random series $\sum_{n\geq 0} X_n z^n$ is almost surely constant.

b) Assume also that the X_n's have the same distribution. Prove that $R = 0$ a.s. if $\mathbb{E}[\log(|X_0|)_+] = \infty$, and $R \geq 1$ a.s. if $\mathbb{E}[\log(|X_0|)_+] < \infty$.

Exercise 8. Let $(X_i)_{1\leq i\leq n}$ be i.i.d. random variables on $(\Omega, \mathcal{A}, \mathbb{P})$, and X_1 uniform on $[0,1]$.

a) With the help of the events $A_\sigma = \{X_{\sigma(1)} < \cdots < X_{\sigma(n)}\}$, $\sigma \in \mathcal{S}_n$, build n random variables $Y_1, \dots, Y_n$ on $(\Omega, \mathcal{A}, \mathbb{P})$ such that $Y_1(\omega) < \cdots < Y_n(\omega)$ a.s. and $\{Y_1(\omega), \dots, Y_n(\omega)\} = \{X_1(\omega), \dots, X_n(\omega)\}$ a.s.

b) Calculate the density of $(Y_1, \dots, Y_n)$ and $(Y_1/Y_2, \dots, Y_{n-1}/Y_n)$.

Exercise 9. Prove that there is no probability measure on $\mathbb{N}$ such that for any $n \geq 1$, the probability of the set of multiples of n is $1/n$.

Exercise 10. Let $(X_n)_{n\geq 0}$ be defined on $(\Omega, \mathcal{A}, \mathbb{P})$. Assume this sequence converges in probability (under $\mathbb{P}$) to X. Let $\mathbb{Q}$ be another probability measure on $(\Omega, \mathcal{A})$ assumed to be absolutely continuous w.r.t. $\mathbb{P}$. Prove that $X_n \to X$ in probability under $\mathbb{Q}$.