Probability, homework 8, due November 23.

Exercise 1. Let \( (S_n)_{n \geq 0} \) be a \( (\mathcal{F}_n) \)-martingale and \( \tau \) a stopping time with finite expectation. Assume that there is a \( c > 0 \) such that, for all \( n \), \( \mathbb{E}(|S_{n+1} - S_n| | \mathcal{F}_n) < c \).

Prove that \( (S_{\tau \wedge n})_{n \geq 0} \) is a uniformly integrable martingale, and that \( \mathbb{E}(S_{\tau}) = \mathbb{E}(S_0) \).

Consider now the random walk \( S_n = \sum_{k=1}^{n} X_k \), the \( X_k \)'s being iid, \( \mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2 \). For some \( a \in \mathbb{N}^* \), let \( \tau = \inf\{n \mid S_n = -a\} \). Prove that
\[
\mathbb{E}(\tau) = \infty.
\]

Exercise 2. As previously, consider the random walk \( S_n = \sum_{k=1}^{n} X_k \), the \( X_k \)'s being iid, \( \mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2 \), \( \mathcal{F}_n = \sigma(X_i, 0 \leq i \leq n) \).

Prove that \( (S_n^2 - n, n \geq 0) \) is a \( (\mathcal{F}_n) \)-martingale. Let \( \tau \) be a bounded stopping time. Prove that \( \mathbb{E}(S_\tau^2) = \mathbb{E}(\tau) \).

Take now \( \tau = \inf\{n \mid S_n \in \{-a, b\}\} \), where \( a, b \in \mathbb{N}^* \). Prove that \( \mathbb{E}(S_\tau) = 0 \) and \( \mathbb{E}(S_\tau^2) = \mathbb{E}(\tau) \). What is \( \mathbb{P}(S_\tau = -a) \)? What is \( \mathbb{E}(\tau) \)? Get the last result of the previous exercise by justifying the limit \( b \to \infty \).

Exercise 3. Let \( X_n, n \geq 0 \), be iid complex random variables such that \( \mathbb{E}(X_1) = 0, 0 < \mathbb{E}(|X_1|^2) < \infty \). For some parameter \( \alpha > 0 \), let
\[
S_n = \sum_{k=1}^{n} \frac{X_k}{k^\alpha}.
\]
Prove that if \( \alpha > 1/2 \), \( S_n \) converges almost surely. What if \( 0 < \alpha \leq 1/2 \) ?

Exercise 4. In a game between a gambler and a croupier, suppose that the total capital in play is 1. After the \( n \)th hand the proportion of the capital held by the gambler is denoted \( X_n \in [0, 1] \), thus that held by the croupier is \( 1 - X_n \). We assume \( X_0 = p \in (0, 1) \). The rules of the game are such that after \( n \) hands, the probability for the gambler to win the \( (n+1) \)th hand is \( X_n \); if he does, he gains half of the capital the croupier held after the \( n \)th hand, while if he loses he gives half of his capital. Let \( \mathcal{F}_n = \sigma(X_i, 1 \leq i \leq n) \).

(i) Show that \( (X_n)_{n \geq 0} \) is a \( (\mathcal{F}_n)_{n \geq 0} \) martingale.
(ii) Show that \( (X_n)_{n \geq 1} \) converges a.s. and in \( L^2 \) towards a limit \( Z \).
(iii) Show that \( \mathbb{E}(X_{n+1}^2) = \mathbb{E}(3X_n^2 + X_n)/4 \). Deduce that \( \mathbb{E}(Z^2) = \mathbb{E}(Z) = p \). What is the law of \( Z \) ?
(iv) For any \( n \geq 0 \), let \( Y_n = 2X_{n+1} - X_n \). Find the conditional law of \( X_{n+1} \) knowing \( \mathcal{F}_n \). Prove that \( \mathbb{P}(Y_n = 0 \mid \mathcal{F}_n) = 1 - X_n \), \( \mathbb{P}(Y_n = 1 \mid \mathcal{F}_n) = X_n \) and express the law of \( Y_n \).
(v) Let \( G_n = \{Y_n = 1\} \), \( P_n = \{Y_n = 0\} \). Prove that \( Y_n \to Z \) a.s. and deduce that \( \mathbb{P}(\liminf_{n \to \infty} G_n) = p \), \( \mathbb{P}(\liminf_{n \to \infty} P_n) = 1 - p \). Are the variables \( \{Y_n, n \geq 1\} \) independent ?
(vi) Interpret the questions (iii), (iv), (v) in terms of gain, loss, for the gambler.

Exercise 5. Let \( a > 0 \) be fixed, \( (X_i)_{i \geq 1} \) be iid, \( \mathbb{R}^d \)-valued random variables, uniformly distributed on the ball \( B(0,a) \). Set \( S_n = x + \sum_{i=1}^{n} X_i \).

(i) Let \( f \) be a superharmonic function. Show that \( (f(S_n))_{n \geq 1} \) defines a supermartingale.
(ii) Prove that if \( d \leq 2 \) any nonnegative superharmonic function is constant. Does this result remain true when \( d \geq 3 \)?

**Exercise 6.** Let \( N_n \) be the size of a population of bacteria at time \( n \). At each time each bacterium produces a number of offspring and dies. The number of offspring is independent for each bacterium and is distributed according to the Poisson law with rate parameter \( \lambda = 2 \). Assuming that \( N_1 = a > 0 \), find the probability that the population will eventually die, i.e., find \( P(\{ \text{there is } n \text{ such that } N_n = 0 \}) \).

**Exercise 7.** Let \( X \) be a standard random walk in dimension 1, and for any positive integer \( a \), \( \tau_a = \inf \{ n \geq 0 \mid X_{\tau_a} = a \} \). For any \( \theta > 0 \), calculate
\[
E(\cosh(\theta) - \tau_a).
\]

**Exercise 8.** Let \( (Y_n)_{n \in \mathbb{N}} \) be a sequence of random variables, and assume \( (Y_n) \) converges in distribution to a limiting \( Y \). Also, on some probability space \( (\Omega, \mathcal{F}, P) \), the sequence of independent random variables \( X := (X_n)_{n \in \mathbb{N}} \) is defined, and we assume that the sequence of partial sums \( (S_n)_{n \in \mathbb{N}} \) (i.e. \( S_0 = 0 \) and \( S_n := \sum_{j=1}^{n} X_j \)) converges in distribution. Set \( (F_n) \) the natural filtration of \( X \) and \( \Phi_n(t) = E(\exp(itS_n)) \) for \( t \in \mathbb{R} \).

(i) Establish that \( (\Phi_{Y_n}(\cdot))_{n \geq 1} \) converges uniformly on every compact, i.e. show that for any \( a > 0 \), \( \max_{t \in [-a,a]} |\Phi_{Y_n}(t) - \Phi_Y(t)| \rightarrow 0 \) as \( n \rightarrow \infty \). Establish moreover that there exists \( a > 0 \) such that for any \( n \geq 1 \), \( \min_{t \in [-a,a]} |\Phi_{Y_n}(t)| \geq 1/2 \).

(ii) Show that there exists \( t_0 > 0 \) such that if \( t \in [-t_0, t_0] \), then \( (\exp(itS_n)/\Phi_n(t))_{n \geq 0} \) is a \( (\mathcal{F}_n) \)-martingale (i.e. both its real and imaginary parts are martingales).

(iii) Prove that we can choose \( t_0 > 0 \) such that for any \( t \in [-t_0, t_0] \), \( \lim_{n \rightarrow \infty} \exp(itS_n) \) exists \( P \)-a.s.

(iv) Set
\[
C = \{(t,\omega) \in [-t_0, t_0] \times \Omega : \lim_{n \rightarrow \infty} \exp(itS_n(\omega)) \text{ exists}\}.
\]
Prove that \( C \) is measurable, i.e. in the product of \( B([-t_0, t_0]) \) with \( \mathcal{F} \).

(v) Establish that \( \int_{-t_0}^{t_0} 1_C(t,\omega)\mathbb{P}(d\omega)dt = 2t_0 \).

(vi) Prove that \( \lim_{n \rightarrow \infty} S_n \) exists \( P \)-a.s.