Probability, homework 1, due September 13.

**Exercise 1.** Let \((\mathcal{G}_\alpha)_{\alpha \in A}\) be an arbitrary family of \(\sigma\)-fields defined on an abstract space \(\Omega\), with \(A\) possibly uncountable. Show that \(\bigcap_{\alpha \in A} \mathcal{G}_\alpha\) is also a \(\sigma\)-field.

**Exercise 2.** Let \(\emptyset \subset A \subset B \subset \Omega\) (these are strict inclusions). What is the \(\sigma\)-field generated by \(\{A, B\}\)?

**Exercise 3.** Let \(\mathcal{F}, \mathcal{G}\) be \(\sigma\)-fields for the same \(\Omega\). Is \(\mathcal{F} \cup \mathcal{G}\) a \(\sigma\)-field?

**Exercise 4.** For \(\Omega = \mathbb{N}\) and \(n \geq 0\), let \(\mathcal{F}_n = \sigma(\{\{0\}, \ldots, \{n\}\})\). Show that \((\mathcal{F}_n)_{n \geq 0}\) is a non-decreasing sequence but that \(\bigcup_{n \geq 0} \mathcal{F}_n\) is not a \(\sigma\)-field.

**Exercise 5.** Let \(\Omega\) be an infinite set (countable or not). Let \(\mathcal{A}\) be the set of subsets of \(\Omega\) that are either finite or with finite complement in \(\Omega\). Prove that \(\mathcal{A}\) is a field but not a \(\sigma\)-field.

**Exercise 6.** Can you build an infinite, countable \(\sigma\)-field?

**Exercise 7.** A monotone class is a collection \(\mathcal{M}\) of sets closed under both monotone increasing and monotone decreasing (i.e. if \(A_i \in \mathcal{M}\) and either \(A_i \uparrow A\) or \(A_i \downarrow A\) then \(A \in \mathcal{M}\)).

Prove that if \(A \subset \mathcal{M}\) with \(A\) a field and \(\mathcal{M}\) a monotone class, then \(\sigma(A) \subset \mathcal{M}\).

**Exercise 8.** Let \(\mathbb{P}\) be a probability measure on \(\Omega\), endowed with a \(\sigma\)-field \(\mathcal{A}\).

(i) What is the meaning of the following events, where all \(A_n\)'s are elements of \(\mathcal{A}\)?

\[
\liminf_{n \to \infty} A_n = \bigcup_{n \geq 1} \bigcap_{k \geq n} A_k, \quad \limsup_{n \to \infty} A_n = \bigcap_{n \geq 1} \bigcup_{k \geq n} A_k.
\]

(ii) Prove that \(\limsup_{n \to \infty} A_n\) and \(\liminf_{n \to \infty} A_n\) are in \(\mathcal{A}\).

(iii) In the special case \(\Omega = \mathbb{R}\), for any \(p \geq 1\), let

\[
A_{2p} = \left[-1, 2 + \frac{1}{2p}\right), \quad A_{2p+1} = \left(-2 - \frac{1}{2p+1}, 1\right].
\]

What are \(\liminf_{n \to \infty} A_n\) and \(\limsup_{n \to \infty} A_n\)?

(iv) Prove that the following always holds:

\[
\mathbb{P}\left(\liminf_{n \to \infty} A_n\right) \leq \liminf_{n \to \infty} \mathbb{P}\left(\limsup_{n \to \infty} A_n\right), \quad \mathbb{P}\left(\limsup_{n \to \infty} A_n\right) \geq \limsup_{n \to \infty} \mathbb{P}\left(\liminf_{n \to \infty} A_n\right).
\]

**Exercise 9.** The symmetric difference of two events \(A\) and \(B\), denoted \(A \Delta B\), is the event that precisely one of them occurs: \(A \Delta B = (A \cup B) \setminus (A \cap B)\).

(i) Write a formula for \(A \Delta B\) that only involves the operations of union, intersection and complement, but no set difference.
(ii) Define $d(A, B) = P(A \triangle B)$. Show that for any three events $A$, $B$, $C$,
\[ d(A, B) + d(B, C) - d(A, C) = 2 \left( P(A \cap B^c \cap C) + P(A^c \cap B \cap C^c) \right). \]

(iii) Assume $A \subset B \subset C$. Prove that $d(A, C) = d(A, B) + d(B, C)$.

**Exercise 10.** Prove the Bonferroni inequalities: if $A_i \in \mathcal{A}$ is a sequence of events, then

(i) $P(\bigcup_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j)$,

(ii) $P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$. 