Probability, homework 1, solutions.

Exercise 1. Let $(\mathcal{G}_\alpha)_{\alpha \in A}$ be an arbitrary family of $\sigma$-fields defined on an abstract space $\Omega$, with $A$ possibly uncountable. Show that $\bigcap_{\alpha \in A} \mathcal{G}_\alpha$ is also a $\sigma$-field.

Solution. We need to check that

(i) $\emptyset \in \bigcap_{\alpha \in A} \mathcal{G}_\alpha$,  
(ii) if $A \in \bigcap_{\alpha \in A} \mathcal{G}_\alpha$, then $A^c \in \bigcap_{\alpha \in A} \mathcal{G}_\alpha$,  
(iii) if $(A_i)_{i \geq 1}$ is a sequence in $\bigcap_{\alpha \in A} \mathcal{G}_\alpha$, then $\bigcup_{i \geq 1} A_i \in \bigcap_{\alpha \in A} \mathcal{G}_\alpha$ and $\bigcap_{i \geq 1} A_i \in \bigcap_{\alpha \in A} \mathcal{G}_\alpha$.

Let us check just the last statement on intersections, all others are proved similarly and left to the reader. If $A_i \in \bigcap_{\alpha \in A} \mathcal{G}_\alpha$, for any $\alpha$ we have $A_i \in \mathcal{G}_\alpha$. As $\mathcal{G}_\alpha$ is a $\sigma$-algebra this implies $\bigcap_{i \geq 1} A_i \in \bigcap_{\alpha \in A} \mathcal{G}_\alpha$. As this is true for any $\alpha$, we have $\bigcap_{i \geq 1} A_i \in \bigcap_{\alpha \in A} \mathcal{G}_\alpha$.

Exercise 2. Let $\emptyset \subset A \subset B \subset \Omega$ (these are strict inclusions). What is the $\sigma$-field generated by $\{A,B\}$?

Solution. It is clear that $\{\emptyset, A, B, A^c, B^c, \Omega\} \subset \sigma(\{A,B\})$. On the other hand, stability of this left hand side by complement, union, intersection can be checked by hand.

Exercise 3. Let $\mathcal{F}, \mathcal{G}$ be $\sigma$-fields for the same $\Omega$. Is $\mathcal{F} \cup \mathcal{G}$ a $\sigma$-field?

Solution. Consider $\mathcal{F} = \{\emptyset, A, A^c, \Omega\}$, $\mathcal{G} = \{\emptyset, B, B^c, \Omega\}$ with $A, B$ nonempty sets such that $A \cup B \neq \Omega$. Then $\mathcal{F} \cup \mathcal{G}$ contains $A, B$ but not $A \cup B$ so it is not a $\sigma$-field.

Exercise 4. For $\Omega = \mathbb{N}$ and $n \geq 0$, let $\mathcal{F}_n = \sigma(\{\{0\}, \ldots, \{n\}\})$. Show that $(\mathcal{F}_n)_{n \geq 0}$ is a non-decreasing sequence but that $\bigcup_{n \geq 0} \mathcal{F}_n$ is not a $\sigma$-field.

Solution. As $\{\{0\}, \ldots, \{n\}\} \subset \{\{0\}, \ldots, \{n+1\}\}$, by definition of $\sigma$ we have $\mathcal{F}_n \subset \mathcal{F}_{n+1}$, i.e. the sequence is non-decreasing.

Assume $\mathcal{A} := \bigcup_{n \geq 0} \mathcal{F}_n$ is a $\sigma$-field. Then the set $A = \{1, 2, \ldots\}$ of all positive integers is in $\mathcal{A}$ because $A = \bigcup_{n \geq 1} [1,n]$ and $[1,n] \in \mathcal{F}_n \subset \mathcal{A}$. Hence there exists $n_0$ such that $A \in \mathcal{F}_{n_0}$. But this contradicts $n_0 + 1 \in A$ and $n_0 + 1$ not contained in any element of $\mathcal{F}_{n_0}$.

Exercise 5. Let $\Omega$ be an infinite set (countable or not). Let $\mathcal{A}$ be the set of subsets of $\Omega$ that are either finite or with finite complement in $\Omega$. Prove that $\mathcal{A}$ is a field but not a $\sigma$-field.

Solution. The set $\mathcal{A}$ is a field: clearly the null and full sets are in it, $\mathcal{A}$ is stable by immediate definition, and if $A$ and $B$ are in $\mathcal{A}$, then $A \cup B$ is either finite (when $A$
and $B$ are finite) or its complement $A^c \cap B^c$ is finite (when $A$ or $B$ is infinite with finite complement), so that $A \cup B \in \mathcal{A}$.

However, $\mathcal{A}$ is not a $\sigma$-field. Indeed, let $\omega_1, \omega_2, \ldots$ be an infinite sequence of disjoint elements in $\Omega$. Then $\{\omega_2, \omega_4, \omega_6, \ldots\}$ is the countable union of elements in $\mathcal{A}$ (the $\{\omega_2i\}$'s), but it is neither finite nor with finite complement, thus it cannot be in $\mathcal{A}$.

**Exercise 6.** Can you build an infinite, countable $\sigma$-field?

**Solution.** Consider $(\Omega, \mathcal{A})$ with $\mathcal{A}$ a $\sigma$-field. For any $x \in \Omega$, define

$$\dot{x} = \bigcap_{A \in \mathcal{A} : x \in A} A.$$ 

Then $\{\dot{x}, x \in \Omega\}$ (eliminating repetitions) form a partition of $\Omega$ (uses stability of the field by complement).

If $\mathcal{A}$ is countable, $\dot{x}$ is an element of $\mathcal{A}$, as a countable intersection of elements in $\mathcal{A}$. Thus $\mathcal{A}$ contains all sets of type $\dot{x}$, and all countable unions of such sets. In fact, $\mathcal{A}$ consists exactly in such unions (easy to prove).

If the number of atoms is finite, then $\mathcal{A}$ is finite. If the number of atoms is infinite, then the number of their countable unions is uncountable (cf a map to Cantor diagonalization), a contradiction.

We have proved that $\mathcal{A}$ is either finite or uncountable.

**Exercise 7.** A monotone class is a collection $\mathcal{M}$ of sets closed under both monotone increasing and monotone decreasing (i.e. if $A_i \in \mathcal{M}$ and either $A_i \uparrow A$ or $A_i \downarrow A$ then $A \in \mathcal{M}$)

Prove that if $\mathcal{A} \subset \mathcal{M}$ with $\mathcal{A}$ a field and $\mathcal{M}$ a monotone class, then $\sigma(\mathcal{A}) \subset \mathcal{M}$.

**Solution.** Clearly, any field which is a monotone class must be a $\sigma$-field. Moreover, we will prove below the following:

(Claim) the intersection $m(\mathcal{A})$ of all monotone classes containing containing a field $\mathcal{A}$ is both a field and a monotone class.

Hence $m(\mathcal{A})$ is a $\sigma$-algebra. Since $\mathcal{A} \subset m(\mathcal{A})$ this implies $\sigma(\mathcal{A}) \subset m(\mathcal{A})$, which concludes the proof by noting $m(\mathcal{A}) \subset \mathcal{M}$.

We now prove the above (Claim). The fact that $m(\mathcal{A})$ is a monotone class is straightforward from the definition. We now show that $m(\mathcal{A})$ is a field. Inclusion of null and full sets is trivial.

For the stability by complement: Consider $\mathcal{B} = \{A : A^c \in m(\mathcal{A})\}$. As $m(\mathcal{A})$ is monotone, so is $\mathcal{B}$. Hence $\mathcal{B}$ is a monotone class containing $\mathcal{A}$, hence $m(\mathcal{A}) \subset \mathcal{B}$, which implies stability by complement.

For stability by union, let

$$\mathcal{G}_1 = \{A : A \cup B \in m(\mathcal{A}) \text{ for all } B \in \mathcal{A}\}.$$ 

Then $\mathcal{G}_1$ is a monotone class (simply checked by careful writing based on definitions) containing $\mathcal{A}$ hence $m(\mathcal{A}) \subset \mathcal{G}_1$. Now let

$$\mathcal{G}_2 = \{B : B \cup A \in m(\mathcal{A}) \text{ for all } A \in m(\mathcal{A})\}.$$
Exercise 8. Let $\mathbb{P}$ be a probability measure on $\Omega$, endowed with a $\sigma$-field $\mathcal{A}$.

(i) What is the meaning of the following events, where all $A_n$'s are elements of $\mathcal{A}$?

\[
\liminf_{n \to \infty} A_n = \bigcup_{n \geq 1} \bigcap_{k \geq n} A_k, \quad \limsup_{n \to \infty} A_n = \bigcap_{n \geq 1} \bigcup_{k \geq n} A_k.
\]

(ii) Prove that $\limsup_{n \to \infty} A_n$ and $\liminf_{n \to \infty} A_n$ are in $\mathcal{A}$.

(iii) In the special case $\Omega = \mathbb{R}$, for any $p \geq 1$, let

\[
A_{2p} = \left[ -1, 2 + \frac{1}{2p} \right), \quad A_{2p+1} = \left( -2 - \frac{1}{2p + 1}, 1 \right].
\]

What are $\liminf_{n \to \infty} A_n$ and $\limsup_{n \to \infty} A_n$?

(iv) Prove that the following always holds:

\[
\mathbb{P} \left( \liminf_{n \to \infty} A_n \right) \leq \liminf_{n \to \infty} \mathbb{P} (A_n), \quad \mathbb{P} \left( \limsup_{n \to \infty} A_n \right) \geq \limsup_{n \to \infty} \mathbb{P} (A_n).
\]

Solution. (i) The event $\liminf_{n \to \infty} A_n$ means that $A_n$ always occurs for large enough $n$. The event $\limsup_{n \to \infty} A_n$ means that infinitely many $A_n$’s happen.

(ii) Elementary by countable intersections and unions.

(iii) For any $n$, $\bigcap_{k \geq n} A_k = [-1, 1]$, so $\liminf_{n \to \infty} A_n = [-1, 1]$. For any $n$, $\bigcup_{k \geq n} A_k = (-2 - a_n, 2 + b_n)$ with $a_n, b_n$ positive converging to 0 as $n \to \infty$, so $\liminf_{n \to \infty} A_n = [-2, 2]$.

(iv) For any $k \geq n$ we have $\mathbb{P}(A_n) \geq \mathbb{P}(\bigcap_{k \geq n} A_k)$ so $\liminf_{n \to \infty} A_n \geq \inf_{n \geq m} \mathbb{P}(\bigcap_{k \geq n} A_k) = \mathbb{P}(\bigcap_{k \geq m} A_k)$. Take the limit $m \to \infty$: The LHS converges to $\liminf \mathbb{P}(A_n)$ by definition of the lim inf, and the RHS converges to $\mathbb{P}(\liminf_{n \to \infty} A_n)$ as the events $\bigcap_{k \geq m} A_k$ increase to $\liminf_{n \to \infty} A_n$.

The proof for lim sup is similar.

Exercise 9. The symmetric difference of two events $A$ and $B$, denoted $A \triangle B$, is the event that precisely one of them occurs: $A \triangle B = (A \cup B) \setminus (A \cap B)$.

(i) Write a formula for $A \triangle B$ that only involves the operations of union, intersection and complement, but no set difference.

(ii) Define $d(A, B) = \mathbb{P}(A \triangle B)$. Show that for any three events $A$, $B$, $C$,

\[
d(A, B) + d(B, C) - d(A, C) = 2 \left( \mathbb{P} (A \cap B^c \cap C) + \mathbb{P} (A^c \cap B \cap C^c) \right).
\]

(iii) Assume $A \subset B \subset C$. Prove that $d(A, C) = d(A, B) + d(B, C)$.

Solution. (i) $A \triangle B = (A \cap B^c) \cup (B \cap A^c)$

(ii) Note that $A \cap B^c$ and $B \cap A^c$ are disjoint so $\mathbb{P}(A \triangle B) = \mathbb{P}(A \cap B^c) + \mathbb{P}(B \cap A^c) = \mathbb{P}(A \cap B^c \cap C) + \mathbb{P}(B \cap A^c \cap C^c) + \mathbb{P}(A \cap B^c \cap C) + \mathbb{P}(B \cap A^c \cap C^c)$. Injecting this formula and the analogues for $\mathbb{P}(A \triangle C)$, $\mathbb{P}(B \triangle C)$ gives the answer.

(iii) Immediate from the previous question as the sets on the RHS are empty.

Exercise 10. Prove the Bonferroni inequalities: if $A_i \in \mathcal{A}$ is a sequence of events, then

(i) $\mathbb{P} (\bigcup_{i=1}^n A_i) \geq \sum_{i=1}^n \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j)$,
(ii) $\Pr (\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} \Pr (A_i) - \sum_{i<j} \Pr (A_i \cap A_j) + \sum_{i<j<k} \Pr (A_i \cap A_j \cap A_k)$.

Solution. For any $x \in \Omega$ we claim that
\[ \mathbf{1}_{x \in \bigcup_{i=1}^{n} A_i} \geq \sum_{i} \mathbf{1}_{x \in A_i} - \sum_{i<j} \mathbf{1}_{x \in A_i \cap A_j}, \]
\[ (0.1) \]
Indeed, if $x$ is in exactly $m$ sets $A_i$’s, for $m = 0$ the above relation is $0 \geq 0$ and if $m \geq 1$ this is
\[ 1 \geq m - \binom{m}{2}, \]
which is true. Integrating (0.1) w.r.t $\Pr$ gives the result.

The same type of reasoning holds for (ii), stopping the binomial expansion at third order.