Probability, homework 2, due September 20.

Exercise 1. Let $\mathcal{A}$ be a $\sigma$-algebra, $\mathbb{P}$ a probability measure and $(A_n)_{n \geq 1}$ a sequence of events in $\mathcal{A}$ which converges to $A$. Prove that

(i) $A \in \mathcal{A}$;
(ii) $\lim_{n \to \infty} \mathbb{P}(A_n) = \mathbb{P}(A)$.

Exercise 2. Suppose a distribution function $F$ is given by

$$F(x) = \frac{1}{4}1_{[0,\infty)}(x) + \frac{1}{2}1_{[1,\infty)}(x) + \frac{1}{4}1_{[2,\infty)}(x)$$

What is the probability of the following events, $(-1/2, 1/2)$, $(-1/2, 3/2)$, $(2/3, 5/2)$, $(3, \infty)$?

Exercise 3. Let $\mu$ be the Lebesgue measure on $\mathbb{R}$. Build a sequence of functions $(f_n)_{n \geq 0}$, $0 \leq f_n \leq 1$, such that $\int f_n \, d\mu \to 0$ but for any $x \in \mathbb{R}$, $(f_n(x))_{n \geq 0}$ does not converge.

Exercise 4. Let $X$ be a random variable in $L^1(\Omega, \mathcal{A}, \mathbb{P})$. Let $(A_n)_{n \geq 0}$ be a sequence of events in $\mathcal{A}$ such that $\mathbb{P}(A_n) \to 0$. Prove that $\mathbb{E}(X1_{A_n}) \to 0$.

Exercise 5. Let $(d_n)_{n \geq 0}$ be a sequence in $(0, 1)$, and $K_0 = [0, 1]$. We define iteratively $(K_n)_{n \geq 0}$ in the following way. From $K_n$, which is the union of closed disjoint intervals, we define $K_{n+1}$ by removing from each interval of $K_n$ an open interval, centered at the middle of the previous one, with length $d_n$ times the length of the previous one. Let $K = \cap_{n \geq 0} K_n$ (K is called a Cantor set).

(a) Prove that $K$ is an uncountable compact set, with empty interior, and whose points are all accumulation points
(b) What is the Lebesgue measure of $K$?

Exercise 6. Let $X$ be a nonnegative random variable. Prove that $\mathbb{E}(X) < +\infty$ if and only if $\sum_{n \in \mathbb{N}} \mathbb{P}(X \geq n) < \infty$.

Exercise 7. Convergence in measure. Let $(\Omega, \mathcal{A}, \mu)$ be a probability space, and $(f_n)_{n \geq 1}, f : \Omega \to \mathbb{R}$ measurable (for the Borel $\sigma$-field on $\mathbb{R}$). We say that $(f_n)_{n \geq 1}$ converges in measure to $f$ if for any $\varepsilon > 0$ we have

$$\mu(|f_n - f| > \varepsilon) \to 0.$$ 

(i) Show that $\int |f - f_n| \, d\mu \to 0$ implies that $f_n$ converges to $f$ in measure. Is the reciprocal true?
(ii) Show that if $f_n \to f$ $\mu$-almost surely, then $f_n \to f$ in measure. Is the reciprocal true?
(iii) Show that if $f_n \to f$ in measure, there exists a subsequence of $(f_n)_{n \geq 1}$ which converges $\mu$-almost surely.
(iv) (A stronger dominated convergence theorem) We assume that $f_n \to f$ in measure and $|f_n| \leq g$ for some integrable $g : \Omega \to \mathbb{R}$, for any $n \geq 1$.

(a) Show that $|f| \leq g$ $\mu$-a.s.
(b) Deduce that $\int |f_n - f| \, d\mu \to 0$.

Exercise 8. Consider a probability space $(\Omega, \mathcal{A}, \mu)$ and $(A_n)_{n}$ a sequence in $\mathcal{A}$. Let $f : \Omega \to \mathbb{R}$ be measurable (for the Borel $\sigma$-field on $\mathbb{R}$) such that $\int_{\Omega} |1_{A_n} - f| \, d\mu \to 0$ as $n \to \infty$. Prove that there exists $A \in \mathcal{A}$ such that $f = 1_A$ $\mu$-a.s., i.e. $\mu(f = 1_A) = 1$. 

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