**Probability, homework 3 due October 4.**

**Exercise 1**. Let $X$ be a random variable with density $f_X(x) = (1-|x|)\mathbb{1}_{(-1,1)}(x)$. Show that its characteristic function is

$$\phi_X(u) = \frac{2(1-\cos u)}{u^2}.$$

**Exercise 2.**

(1) Prove that $\hat{\mu}$ is real-valued if and only if $\mu$ is symmetric, i.e. $\mu(A) = \mu(-A)$ for any Borel set $A$

(2) If $X$ and $Y$ are i.i.d., prove that $X - Y$ has a symmetric distribution.

**Exercise 3**. Let $X_\lambda$ be a real random variable, with Poisson distribution with parameter $\lambda$. Calculate the characteristic function of $X_\lambda$. Conclude that $(X_\lambda - \lambda)/\sqrt{\lambda}$ converges in distribution to a standard Gaussian, as $\lambda \to \infty$.

**Exercise 4.** Assume that the sequence of random variables $(X_n)_{n\geq 1}$ satisfies $\mathbb{E}\, X_n \to 1$ and $\mathbb{E}\, X_n^2 \to 1$. Prove that $(X_n)_{n\geq 1}$ converges in distribution. What is the limit?

**Exercise 5**. Let $(X_n)_{n\geq 1}$, $(Y_n)_{n\geq 1}$ be real random variables, with $X_n$ and $Y_n$ independent for any $n \geq 1$, and assume that $X_n$ converges in distribution to $X$ and $Y_n$ to $Y$, with $X$ and $Y$ independent defined on the same probability space. Prove that $X_n + Y_n$ converges in distribution to $X + Y$.

**Exercise 6**. Let $X, Y$ be independent and assume that for some constant $\alpha$ we have $\mathbb{P}(X + Y = \alpha) = 1$. Prove that $X$ and $Y$ are both constant random variables.

**Exercise 7.** Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with standard Cauchy distribution and let $M_n = \max(X_1, \dots, X_n)$. Prove that $(nM_n^{-1})_{n\geq 1}$ converges in distribution and identify the limit.

**Exercise 8** Let $X$, $Y$ be i.i.d., with characteristic functions denoted $\varphi_X, \varphi_Y$, and suppose $\mathbb{E}(X) = 0$, $\mathbb{E}(X^2) = 1$. Assume also that $X+Y$ and $X-Y$ are independent.

(1) Prove that

$$\varphi_X(2u) = (\varphi_X(u))^3 \varphi_X(-u)$$

(2) Prove that $X$ is a standard Gaussian random variable.

**Exercise 9**. For any $d \geq 1$, we admit that there is only one probability measure $\mu$ on $\mathcal{S}_d$, (the $(d-1)$-th dimensional sphere embedded in $\mathbb{R}^d$) that is uniform, in the following sense: for any isometry $A \in \mathrm{O}(d)$ (the orthogonal group in $\mathbb{R}^d$), and any continuous function $f : \mathcal{S}_d \to \mathbb{R}$,

$$\int_{\mathcal{S}_d} f(x)\mathrm{d}\mu(x) = \int_{\mathcal{S}_d} f(Ax)\mathrm{d}\mu(x).$$

Let $X = (X_1, \dots, X_d)$ be a vector of independent centered and reduced Gaussian random variables.

a) Prove that the random variable $U = X/\|X\|_{L^2}$ is uniformly distributed on the sphere.

b) Prove that, as $d \to \infty$, the main part of the globe is concentrated close to the Equator, i.e. for any $\varepsilon > 0$,

$$\int_{x\in\mathcal{S}_d, |x_1|<\epsilon} \mathrm{d}\mu(x) \to 1.$$