Exercise 1. Prove that if a sequence of real random variables \((X_n)\) converge in distribution to \(X\), and \((Y_n)\) converges in distribution to a constant \(c\), then \(X_n + Y_n\) converges in distribution to \(X + c\).

Exercise 2. Assume that \((X, Y)\) has joint density
\[
e^{-((1+x^2)(1+y^2))},
\]
where \(c\) is properly chosen. Are \(X\) and \(Y\) Gaussian random variables? Is \((X, Y)\) a Gaussian vector?

Exercise 3. Let \((X_i)_{i \geq 1}\) be a sequence of independent random variables, with \(X_i\) uniform on \([-i, i]\). Let \(S_n = X_1 + \cdots + X_n\). Prove that \(S_n/n^{3/2}\) converges in distribution and describe the limit.

Exercise 4. Find a probability distribution \(\mu\) of a \(Z\)-valued random variable which is symmetric (\(\mu(\{i\}) = \mu(\{-i\})\) for any \(i \in \mathbb{Z}\)), not integrable, but such that its characteristic function is differentiable at 0.

Exercise 5. For any probability measure \(\mu\) supported on \(\mathbb{R}_+\), one defines the Laplace transform as
\[
L_\mu(\lambda) = \int_0^\infty e^{-\lambda x} d\mu(x), \quad \lambda \geq 0.
\]
(1) Prove that \(L_\mu\) is well-defined, continuous on \(\mathbb{R}_+\) and \(\mathcal{C}_\infty\) on \(\mathbb{R}_+^*\).
(2) Prove that \(L_\mu\) characterizes the probability measure \(\mu\) supported on \(\mathbb{R}_+\).
(3) Assume that for a sequence \((\mu_n)_{n \geq 1}\) of probability measure supported on \(\mathbb{R}_+\), one has \(L_{\mu_n}(\lambda) \to \ell(\lambda)\) for any \(\lambda \geq 0\), and \(\ell\) is right-continuous at 0. Prove that \((\mu_n)_{n \geq 1}\) is tight, and that it converges weakly to a measure \(\mu\) such that \(\ell = L_\mu\).

Long problem. The goal of this problem is to prove the iterated logarithm law, first for Gaussian random variables. In other words, for \(X_1, X_2 \ldots\) i.i.d. standard Gaussian random variables, denoting \(S_n = X_1 + \cdots + X_n\), we have
\[
P \left( \lim sup_{n \to \infty} \frac{S_n}{\sqrt{2n \log \log n}} = 1 \right) = 1 \tag{0.1}
\]
(1) Prove that
\[
P(X_1 > \lambda) \sim \frac{1}{\lambda \sqrt{2\pi}} e^{-\frac{\lambda^2}{2}}.
\]
In the following questions we denote \(f(n) = \sqrt{2n \log \log n}, \lambda > 1, c, \alpha > 0, A_k = \{S_{[\lambda^k]} \geq cf(\lambda^k)\}', C_k = \{S_{[\lambda^{k+1}]} - S_{[\lambda^k]} \geq cf(\lambda^{k+1} - \lambda^k)\}'\) and \(D_k = \{\sup_{n \in [\lambda^k, \lambda^{k+1}]} f(\lambda^k) \geq \alpha\}\).
(2) Prove that for any \(c > 1\) we have \(\sum_{k \geq 1} P(A_k) < \infty\) and
\[
\lim sup_{k \to \infty} \frac{S_{[\lambda^k]}}{f(\lambda^k)} \leq 1 \text{ a.s.}
\]
(3) Prove that for any \( c < 1 \) we have \( \sum_{k \geq 1} \mathbb{P}(C_k) = \infty \) and
\[
\mathbb{P}(C_k \text{ i.o.}) = 1.
\]
(4) Let \( \varepsilon > 0 \) and choose \( c = 1 - \varepsilon/10 \). Prove that almost surely the following inequality holds for infinitely many \( k \):
\[
\frac{S_{\lfloor \lambda k + 1 \rfloor}}{f(\lambda k + 1)} \geq c \frac{f(\lambda k + 1) - \lambda^k}{f(\lambda k + 1)} - (1 + \varepsilon) \frac{f(\lambda k)}{f(\lambda k + 1)}.
\]
(5) By choosing a large enough \( \lambda \) in the previous inequality, prove that almost surely
\[
\limsup_{n \to \infty} \frac{S_n}{f(n)} \geq 1.
\]
(6) Prove that for any \( n \in \lfloor \lambda k, \lambda k + 1 \rfloor \) and \( S_n > 0 \) we have
\[
\frac{S_n}{f(n)} \leq \frac{S_{\lfloor \lambda k \rfloor}}{f(\lfloor \lambda k \rfloor)} + \frac{S_n - S_{\lfloor \lambda k \rfloor}}{f(\lfloor \lambda k \rfloor)}.
\]
(7) Prove that
\[
\mathbb{P}(D_k) \sim \frac{2}{k \to \infty} \mathbb{P}
\left( X_1 \geq \frac{\alpha f(\lambda k)}{\sqrt{\lambda k + 1 - \lambda^k}} \right) \sim \frac{c}{\sqrt{\log \lambda}} \left( \frac{1}{k} \right)^{\alpha^2}.
\]
(8) Prove that for \( \alpha^2 > \lambda - 1 \), almost surely
\[
\limsup_{n \to \infty} \frac{S_n}{f(n)} \leq \limsup_{n \to \infty} \frac{S_{\lfloor \lambda k \rfloor}}{f(\lambda k)} + \alpha.
\]
(9) By choosing appropriate \( \lambda \) and \( \alpha \), prove that almost surely
\[
\limsup_{n \to \infty} \frac{S_n}{f(n)} \leq 1.
\]
(10) State a result similar to (0.1) for i.i.d. uniformly bounded random variables. Which steps in the above proof need to be modified to prove this universality result? How?