Exercise 1. Assume $(\Omega, \mathcal{A}, \mathbb{P})$ is such that $\Omega$ is countable and $\mathcal{A} = 2^\Omega$. Prove that convergence in probability and convergence almost sure are the same.

Exercise 2. Let $(X_i)_{i \geq 1}$ be i.i.d. Gaussian with mean 1 and variance 3. Show that
\[
\lim_{n \to \infty} \frac{X_1 + \cdots + X_n}{\sqrt{X_1^2 + \cdots + X_n^2}} = \frac{1}{4} \text{ a.s.}
\]

Exercise 3. Let $f$ be a continuous function on $[0, 1]$. Calculate the asymptotics, as $n \to \infty$, of
\[
\int_{[0,1]^n} f \left( \frac{x_1 + \cdots + x_n}{n} \right) \, dx_1 \ldots dx_n.
\]

Exercise 4. The goal of this exercise is to prove that any function, continuous on an interval of $\mathbb{R}$, can be approximated by polynomials, arbitrarily close for the $L^\infty$ norm (this is the Bernstein-Weierstrass theorem). Let $f$ be a continuous function on $[0, 1]$. The $n$-th Bernstein polynomial is
\[
B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f \left( \frac{k}{n} \right).
\]

a) Let $S_n(x) = B^{(n,x)}/n$, where $B^{(n,x)}$ is a binomial random variable with parameters $n$ and $x$: $B^{(n,x)} = \sum_{i=1}^{n} X_i$ where the $X_i$’s are independent and $\mathbb{P}(X_i = 1) = x, \mathbb{P}(X_i = 0) = 1-x$. Prove that $B_n(x) = E(f(S_n(x)))$.

b) Prove that $\|B_n - f\|_{L^\infty([0,1])} \to 0$ as $n \to \infty$.

Exercise 5. Calculate
\[
\lim_{n \to \infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}.
\]

Exercise 6. Let $\alpha > 0$ and, given $(\Omega, \mathcal{A}, \mathbb{P})$, let $(X_n, n \geq 1)$ be a sequence of independent real random variables with law $\mathbb{P}(X_n = 1) = \frac{1}{n^\alpha}$ and $\mathbb{P}(X_n = 0) = 1 - \frac{1}{n^\alpha}$. Prove that $X_n \to 0$ in $\mathcal{L}^1$, but that almost surely
\[
\limsup_{n \to \infty} X_n = \begin{cases} 
1 & \text{if } \alpha \leq 1 \\
0 & \text{if } \alpha > 1
\end{cases}.
\]

Exercise 7. A sequence of random variables $(X_i)_{i \geq 1}$ is said to be completely convergent to $X$ if for any $\varepsilon > 0$, we have $\sum_{i \geq 1} \mathbb{P}(|X_i - X| < \varepsilon) < \infty$. Prove that complete convergence implies almost sure convergence.

Exercise 8. Let $(X_n)_{n \geq 1}$ be a sequence of random variables, on the same probability space, with $E(X_\ell) = \mu$ for any $\ell$, and a weak correlation in the following sense: $\text{Cov}(X_k, X_\ell) \leq f(|k - \ell|)$ for all indexes $k, \ell$, where the sequence $(f(m))_{m \geq 0}$ converges to 0 as $m \to \infty$. Prove that $(n^{-1} \sum_{k=1}^{n} X_k)_{n \geq 1}$ converges to $\mu$ in $\mathcal{L}^2$.

Long problem. The goal is to prove the Erdős-Kac theorem: if $w(m)$ denotes the number of distinct prime factors of $m$ and $k$ is a random variable uniformly distributed on $[1, n]$, then the following convergence in distribution holds:
\[
\frac{w(k) - \log \log n}{\sqrt{\log \log n}} \xrightarrow{n \to \infty} \mathcal{N}(0, 1).
\]

(i) Prove that if $(X_n)_{n \geq 1}$ converges in distribution to $\mathcal{N}(0, 1)$ and $\sup_{n \geq 1} \mathbb{E}[X_n^{2k}] < \infty$ for any $k \in \mathbb{N}$, then $\lim_{n \to \infty} \mathbb{E}[X_n^k] = \mathbb{E}[\mathcal{N}(0, 1)^k]$ for any $k \in \mathbb{N}$.
(ii) Prove that for any $x \in \mathbb{R}$ and $d \geq 1$ we have
\[
\left| e^{ix} - \sum_{\ell=0}^{d} \frac{(ix)^{\ell}}{\ell!} \right| \leq \frac{|x|^{d+1}}{(d+1)!}.
\]

(iii) Assume that $\lim_{n \to \infty} \mathbb{E}[X_n^k] = \mathbb{E}[\mathcal{N}(0,1)^k]$ for any $k \in \mathbb{N}$. Prove that $X_n$ converges in distribution to $X$.

(iv) Let $w_y(m)$ be the number of prime factors of $m$ which are smaller than $y$.
Let $(B_p)_{p \text{ prime}}$ be independent random variables such that $\mathbb{P}(B_p = 1) = 1 - \mathbb{P}(B_p = 0) = \frac{1}{p}, \ W_y = \sum_{p \leq y} B_p, \ \mu_y = \sum_{p \leq y} \frac{1}{p}, \ \sigma^2_y = \sum_{p \leq y} \left( \frac{1}{p} - \frac{1}{p^2} \right)$.
Prove that if $y = n^{o(1)}$, then for any $d \in \mathbb{N}$ we have
\[
\mathbb{E} \left[ \left( \frac{w_y(k) - \mu_y}{\sigma_y} \right)^d \right] - \mathbb{E} \left[ \left( \frac{W_y - \mu_y}{\sigma_y} \right)^d \right] \to 0
\]
as $n \to \infty$.

(v) Conclude.