Solution. We already know that convergence a.s. implies convergence in probability. Assume now that $X_n \to X$ in probability.

As $A = 2^\Omega$, the set $A$ of $\omega$'s such that $\mathbb{P}(\{\omega\}) > 0$ is measurable. We need to prove that the measure of $\omega$'s such that $X_n(\omega) \to X(\omega)$ is 1, which is equivalent to showing that the desired asymptotics is $\mathbb{P}(\{\omega\}) = 1$.

From convergence in probability, for any $\varepsilon > 0$ we have

$$\sum_{\omega \in A} \mathbb{P}(\{\omega\}) \mathbb{I}_{|X_n(\omega) - X(\omega)| > \varepsilon} \to 0$$

as $N \to \infty$. In particular, for any $\omega \in A$, $\mathbb{I}_{|X_n(\omega) - X(\omega)| > \varepsilon} \to 0$. As this is true for any $\varepsilon$, this means that for any $\omega \in A$ we have $X_n(\omega) \to X(\omega)$, which concludes the proof.

Exercise 2. Let $(X_i)_{i \geq 1}$ be i.i.d. Gaussian with mean 1 and variance 3. Show that

$$\lim_{n \to \infty} \frac{X_1 + \ldots + X_n}{\sqrt{n}} = \frac{1}{4}$$

a.s.

Solution. Let $A = \{\lim_{n \to \infty} \frac{X_1 + \ldots + X_n}{n} = 1\}$ and $B = \{\lim_{n \to \infty} \frac{X_1^2 + \ldots + X_n^2}{n} = 4\}$. The $X_i$'s and $X_i^2$'s are in $L^1$ and $\mathbb{E}(X_i) = 1$, $\mathbb{E}(X_i^2) = 3 + \mathbb{E}(X_i)^2 = 4$, so by the law of large numbers $\mathbb{P}(A) = \mathbb{P}(B) = 1$, so that $\mathbb{P}(A \cap B) = 1$. As $A \cap B \subset \{\lim_{n \to \infty} \frac{X_1 + \ldots + X_n}{\sqrt{n}} = \frac{1}{4}\}$, the result follows.

Exercise 3. Let $f$ be a continuous function on $[0, 1]$. Calculate the asymptotics, as $n \to \infty$, of

$$\int_{[0,1]^n} f \left( \frac{x_1 + \ldots + x_n}{n} \right) \, dx_1 \ldots dx_n.$$

Solution. Let $U_1, U_2, \ldots$ be independent random variables, uniform on $[0, 1]$. By the law of large numbers, $(U_1 + \ldots + U_n)/n \to 1/2$ almost surely, hence in distribution. This implies that for any continuous function on $[0, 1]$ (and therefore bounded),

$$\mathbb{E} f \left( \frac{U_1 + \ldots + U_n}{n} \right) \to f \left( \frac{1}{2} \right),$$

which is equivalent to showing that the desired asymptotics is $f \left( \frac{1}{2} \right)$.

Exercise 4. The goal of this exercise is to prove that any function, continuous on an interval of $\mathbb{R}$, can be approximated by polynomials, arbitrarily close for the $L^\infty$ norm (this is the Bernstein-Weierstrass theorem). Let $f$ be a continuous function on $[0, 1]$. The $n$-th Bernstein polynomial is

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f \left( \frac{k}{n} \right).$$

a) Let $S_n(x) = B^{(n,x)}/n$, where $B^{(n,x)}$ is a binomial random variable with parameters $n$ and $x$: $B^{(n,x)} = \sum_{i=1}^n X_i$ where the $X_i$'s are independent and $\mathbb{P}(X_i = 1) = x$, $\mathbb{P}(X_i = 0) = 1 - x$. Prove that $B_n(x) = \mathbb{E}(f(S_n(x)))$.

b) Prove that $\|B_n - f\|_{L^\infty([0,1])} \to 0$ as $n \to \infty$.

Solution.
(a) We have $P(B^{(n,x)} = k) = \binom{n}{k} x^k (1-x)^{n-k}$, so that
\[
E(f(S_n(x))) = \sum_{k=0}^{n} P(B^{(n,x)} = k) \cdot f\left(\frac{k}{n}\right) = B_n(x).
\]

(b) Let $\varepsilon > 0$. As $f$ is continuous on the compact $[0,1]$, it is uniformly continuous, i.e. there exists $\delta > 0$ such that $|f(a) - f(b)| < \varepsilon$ for any $a, b \in [0,1]$ satisfying $|a - b| < \delta$. Then
\[
|B_n(x) - f(x)| = |E(f(S_n(x)) - f(x))| \\
\leq |E((f(S_n(x)) - f(x))\mathbf{1}_{|S_n(x) - x| < \delta})| + |E((f(S_n(x)) - f(x))\mathbf{1}_{|S_n(x) - f(x)| \geq \delta})| \\
\leq \varepsilon + 2\|f\|_{\infty} P(|S_n(x) - x| > \delta) \leq \varepsilon + 2\|f\|_{\infty} \delta^{-2} E[(S_n(x) - x)^2].
\]

We have $E[(S_n(x) - x)^2] = n^{-2} Var(\sum X_i) = n^{-2} \sum Var(X_i) = n^{-1} Var(X_1)$, so that for $n > 2\|f\|_{\infty} \delta^{-2}$ we obtain $|B_n(x) - f(x)| \leq 2\varepsilon$, which concludes the proof.

**Exercise 5.** Calculate
\[
\lim_{n \to \infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}.
\]

**Solution.** Let $X_n$ be a Poisson random variable with parameter $n$. Then the sum of interest can exactly be interpreted as
\[
P(X_n \leq n) = P\left(\frac{X_n - n}{\sqrt{n}} \leq 0\right).
\]
From an exercise in a previous homework, we know this converges (as $n \to \infty$) to $P(\mathcal{N}(0,1) \leq 0) = \frac{1}{2}$.

**Exercise 6.** Let $\alpha > 0$ and, given $(\Omega,\mathcal{A},P)$, let $(X_n, n \geq 1)$ be a sequence of independent real random variables with $\mathbb{P}(X_n = 1) = \frac{1}{n^\alpha}$ and $\mathbb{P}(X_n = 0) = 1 - \frac{1}{n^\alpha}$. Prove that $X_n \to 0$ in $\mathcal{L}^1$, but that almost surely
\[
\limsup_{n \to \infty} X_n = \begin{cases} 
1 & \text{if } \alpha \leq 1 \\
0 & \text{if } \alpha > 1
\end{cases}.
\]

**Solution.** For convergence to $0$ in $\mathcal{L}^1$, we just write
\[
E(\|X_n\|) = n^{-\alpha} \to 0.
\]
If $\alpha > 1$, then $\sum P(X_n = 1) < \infty$ so by Borel Cantelli $\mathbb{P}(X_n = 1 \text{ i.o.}) = 0$, so $\mathbb{P}(\limsup_{n \to \infty} X_n = 0) = 1$.
If $\alpha \leq 1$, the reverse direction (which requires independent $X_n$’s) of Borel Cantelli also applies in a similar way.

**Exercise 7.** A sequence of random variables $(X_i)_{i \geq 1}$ is said to be completely convergent to $X$ if for any $\varepsilon > 0$, we have $\sum_{i \geq 1} \mathbb{P}(|X_i - X| > \varepsilon) < \infty$. Prove that if the $X_i$’s are independent then complete convergence implies almost sure convergence.

**Solution.** By Borel Cantelli, under the assumption of complete convergence, for any $\varepsilon > 0$ we have
\[
\mathbb{P}(|X_i - X| > \varepsilon \text{ i.o.}) = 0.
\]
This implies
\[ P(\lim \sup (X_i - X) > \varepsilon) = P(\lim \inf (X_i - X) < -\varepsilon) = 0. \]
By monotonicity of the sets, taking \( \varepsilon \to 0 \) in the above convergence we get
\[ P(\lim \sup (X_i - X) > 0) = P(\lim \inf (X_i - X) < 0) = 0, \]
so that \( P(\lim (X_i - X) = 0) = 1. \)
Note that we have not used the independence of the \( X_i \)'s. It could be used to establish the reciprocal.

**Exercise 8.** Let \((X_n)_{n \geq 1}\) be a sequence of random variables, on the same probability space, with \( E(X_\ell) = \mu \) for any \( \ell \), and a weak correlation in the following sense: \( \text{Cov}(X_k, X_\ell) \leq f(|k - \ell|) \) for all indexes \( k, \ell \), where the sequence \((f(m))_{m \geq 0}\) converges to 0 as \( m \to \infty \). Prove that \((n^{-1} \sum_{k=1}^n X_k)_{n \geq 1}\) converges to \( \mu \) in \( L^2 \).

**Solution.** Without loss of generality we can assume \( \mu = 0 \). Then expansion gives
\[ E \left( (n^{-1} \sum_{i=1}^n X_i)^2 \right) \leq n^{-2} \sum_{1 \leq k, \ell \leq n} f(|k - \ell|) \leq n^{-1} \sum_{0 \leq i \leq n} |f(i)| \]
As \( f \) converges to 0, the above RHS also converges to 0, which concludes the proof.

**Long problem.** The goal is to prove the Erdős-Kac theorem: if \( w(m) \) denotes the number of distinct prime factors of \( m \) and \( k \) is a random variable uniformly distributed on \([1, n]\), then the following convergence in distribution holds:
\[ \frac{w(k) - \log \log n}{\sqrt{\log \log n}} \xrightarrow{n \to \infty} \mathcal{N}(0, 1). \]

(i) Prove that if \((X_n)_{n \geq 1}\) converges in distribution to \( \mathcal{N}(0, 1) \) and \( \sup_{n \geq 1} E[X_n^{2k}] < \infty \) for any \( k \in \mathbb{N} \), then \( \lim_{n \to \infty} E[X_n^k] = E[\mathcal{N}(0, 1)^k] \) for any \( k \in \mathbb{N} \).

(ii) Prove that for any \( x \in \mathbb{R} \) and \( d \geq 1 \) we have
\[ \left| e^{ix} - \sum_{\ell=0}^{d} \frac{(ix)^\ell}{\ell!} \right| \leq \frac{|x|^{d+1}}{(d+1)!}. \]

(iii) Assume that \( \lim_{n \to \infty} E[X_n^k] = E[\mathcal{N}(0, 1)^k] \) for any \( k \in \mathbb{N} \). Prove that \( X_n \) converges in distribution to \( X \).

(iv) Let \( w_y(m) \) be the number of prime factors of \( m \) which are smaller than \( y \). Let \((B_p)_{p \text{ prime}}\) be independent random variables such that \( P(B_p = 1) = 1 - P(B_p = 0) = \frac{1}{p} \), \( W_y = \sum_{p \leq y} B_p \), \( \mu_y = \sum_{p \leq y} \frac{1}{p} \), \( \sigma_y^2 = \sum_{p \leq y} (\frac{1}{p} - \frac{1}{p^2}) \).
Prove that if \( y = n^{o(1)} \), then for any \( d \in \mathbb{N} \) we have
\[ E \left[ \left( \frac{w_y(k) - \mu_y}{\sigma_y} \right)^d \right] - E \left[ \left( \frac{W_y - \mu_y}{\sigma_y} \right)^d \right] \to 0 \]
as \( n \to \infty \).

(v) Conclude.

**Solution.**
(i) For $M > 0$, let $f_M(x) = x^k$ if $|x| < M$, $M$ otherwise. Then
\[
\left| \mathbb{E}[X_n^k - \mathcal{N}(0, 1)^k] \right| \leq \left| \mathbb{E}[f_M(X_n) - f_M(\mathcal{N}(0, 1))] \right| + \left| \mathbb{E}[|X_n^k| - |\mathcal{N}(0, 1)|^k] \right| + \left| \mathbb{E}[|\mathcal{N}(0, 1)^k| - |\mathcal{N}(0, 1)|^k] \right|
\]
\[
\leq \left| \mathbb{E}[f_M(X_n) - f_M(\mathcal{N}(0, 1))] \right| + \left| \mathbb{E}[X_n^k/M^k] \right| + \left| \mathbb{E}[\mathcal{N}(0, 1)^k/M^k] \right|
\]
\[
\leq \left| \mathbb{E}[f_M(X_n) - f_M(\mathcal{N}(0, 1))] \right| + C_k/M^k.
\]

For fixed $\varepsilon > 0$, one can find $M > 0$ such that $C_k/M^k < \varepsilon$. For this $M$, by weak convergence the expectation above converges to 0. Hence for large enough $n$ we have $|\mathbb{E}[X_n^k - \mathcal{N}(0, 1)^k]| \leq 2\varepsilon$.

(ii) This is an easy exercise from the Taylor formula with integral remainder.

(iii) Let $u \in \mathbb{R}$ be fixed. One easily proves that $\mathbb{E}[|\mathcal{N}(0, 1)^2| u 2^k] \ll (2\ell)!$. Hence by the previous question for any $\varepsilon > 0$ we can find $d > 0$ such that
\[
\left| \mathbb{E}[e^{iuX_n} - e^{iu\mathcal{N}(0, 1)}] \right| \leq \sum_{\ell=0}^{d} \left| \mathbb{E}[X_n^\ell - \mathcal{N}(0, 1)^\ell] \right| + \varepsilon.
\]
Convergence of the moments then gives $\left| \mathbb{E}[e^{iuX_n} - e^{iu\mathcal{N}(0, 1)}] \right| \leq 2\varepsilon$ for large enough $n$, which concludes the proof.

(iv) Note that, by binomial expansion,
\[
\mathbb{E}[(w_y - \mu_y)^d] - \mathbb{E}[(W_y - \mu_y)^d] = \sum_{j=0}^{d} \binom{d}{j} (-\mu_y)^{d-j} (\mathbb{E}[(w_y(k))^j] - \mathbb{E}[(W_y)^j]),
\]
thus we only need to prove that for any fixed $j$,
\[
\mathbb{E}[(w_y(k))^j] - \mathbb{E}[(W_y)^j] = o(\mu_y^{j-\frac{d}{2}}).
\]
We have
\[
\mathbb{E}[(w_y(k))^j] = \sum_{p_1, \ldots, p_j \leq y} \mathbb{P}(p_1 | k, \ldots, p_j | k) = \sum_{p_1, \ldots, p_j \leq y} \frac{n/lcm(p_1, \ldots, p_j)}{n}
\]
\[
= \sum_{p_1, \ldots, p_j \leq y} \frac{1}{lcm(p_1, \ldots, p_j)} + O(\pi(y)^j/n) = \mathbb{E}[(W_y)^j] + O(y^j/n) = \mathbb{E}[(W_y)^j] + o(\mu_y^{j-\frac{d}{2}}),
\]
concluding the proof.

(v) We now choose $y = \lfloor n^{1/\log\log\log n} \rfloor$. From the previous questions, $\frac{w_y(k) - \mu_y}{\sigma_y}$ converges to a standard Gaussian. Any $k \leq n$ has at most $\log n/\log y = \log\log\log n$ prime factors in $[y, n]$, we have $w_y(k) = w(k) + O(\log\log\log n)$.

Moreover a calculation gives $\mu_y = \log y + O(1) = \log n + O(\log\log\log n)$ and $\sigma_y^2 \sim \log\log n$, so that $\frac{w_y(k) - \mu_y}{\sigma_y}$ also converges in distribution to a standard Gaussian. Injecting the estimates on $\mu_y$ and $\sigma_y$ concludes the proof.