Exercise 1. Let $\mathcal{A}$ be a $\sigma$-algebra, $\mathbb{P}$ a probability measure and $(A_n)_{n \geq 1}$ a sequence of events in $\mathcal{A}$ which converges to $A$. Prove that

(i) $A \in \mathcal{A}$;
(ii) $\lim_{n \to \infty} \mathbb{P}(A_n) = \mathbb{P}(A)$.

Solution. As a first step, note that $A_n \to A$ can be rephrased as

$$A = \limsup_{n \to \infty} A_n = \liminf_{n \to \infty} A_n,$$

where we remind that $\limsup A_n = \cap_{n \geq 1} \cup_{m \geq n} A_m$ and $\liminf A_n = \cup_{n \geq 1} \cap_{m \geq n} A_m$. In particular this implies that $A \in \mathcal{A}$.

We therefore known that $(B_n)_n := (\cup_{m \geq n} A_m)_n$ decreases to $A$ and $(C_n)_n := (\cap_{m \geq n} A_m)_n$ increases to $A$, so $\mathbb{P}(B_n) \to \mathbb{P}(A)$ and $\mathbb{P}(C_n) \to \mathbb{P}(A)$. As $B_n \subset A_n \subset C_n$, this implies $\mathbb{P}(A_n) \to \mathbb{P}(A)$.

Exercise 2. Suppose a distribution function $F$ is given by

$$F(x) = \frac{1}{4} \mathbb{I}_{[0, \infty)}(x) + \frac{1}{2} \mathbb{I}_{[1, \infty)}(x) + \frac{1}{4} \mathbb{I}_{[2, \infty)}(x).$$

What is the probability of the following events, $(-1/2, 1/2), (-1/2, 3/2), (2/3, 5/2), (3, \infty)$?

Solution We have

$$\mathbb{P}((-1/2, 1/2)) = F(1/2) - F(-1/2) = 1/4 - 0 = 1/4,$$
$$\mathbb{P}((-1/2, 3/2)) = F(3/2) - F(-1/2) = 3/4 - 0 = 3/4,$$
$$\mathbb{P}((2/3, 5/2)) = F(5/2) - F(2/3) = 1 - 1/4 = 3/4,$$
$$\mathbb{P}((3, \infty)) = F(\infty) - F(3) = 1 - 1 = 0.$$

Exercise 3. Let $\mu$ be the Lebesgue measure on $\mathbb{R}$. Build a sequence of functions $(f_n)_{n \geq 0}$, $0 \leq f_n \leq 1$, such that $\int f_n \, d\mu \to 0$ but for any $x \in \mathbb{R}$, $(f_n(x))_{n \geq 0}$ does not converge.

Solution Define $f_n(x) = \mathbb{I}_{x \in [f(n), g(n)]}$ where, for $n \in [2^p, 2^{p+1})$, $f(n) = (n - 2^p - 2^{p-1})/p$, $g(n) = f(n) + 1/p$. Clearly $\int f_n = 1/p \to 0$ and for any $x$ we have $f_n(x) = 1$, and 0 i.o.

Exercise 4. Let $X$ be a random variable in $L^1(\Omega, \mathcal{A}, \mathbb{P})$. Let $(A_n)_{n \geq 0}$ be a sequence of events in $\mathcal{A}$ such that $\mathbb{P}(A_n) \to 0$. Prove that $\mathbb{E}(X \mathbb{1}_{A_n}) \to 0$.

Solution For any $C > 0$ we have

$$|\mathbb{E}(X \mathbb{1}_{A_n})| \leq \mathbb{E}(|X| \mathbb{1}_{|X| > C}) + C \mathbb{P}(A_n).$$

Let $\varepsilon > 0$. By monotone convergence, there exists $C > 0$ such that $\mathbb{E}(|X| \mathbb{1}_{|X| > C}) < \varepsilon$. For this $C$, there exists a $n_0$ such that for any $n > n_0$ we have $\mathbb{P}(A_n) < \varepsilon/C$. Thus we have proved that for $n > n_0$, $|\mathbb{E}(X \mathbb{1}_{A_n})| < 2\varepsilon$, which concludes the proof.

Exercise 5. Let $(d_n)_{n \geq 0}$ be a sequence in $(0,1)$, and $K_0 = [0,1]$. We define iteratively $(K_n)_{n \geq 0}$ in the following way. From $K_n$, which is the union of closed disjoint intervals, we define $K_{n+1}$ by removing from each interval of $K_n$ an open interval, centered at the middle of the previous one, with length $d_n$ times the length of the previous one. Let $K = \cap_{n \geq 0} K_n$ (K is called a Cantor set).
(a) Prove that \( K \) is an uncountable compact set, with empty interior, and whose points are all accumulation points.
(b) What is the Lebesgue measure of \( K \)?

Solution. (i) Each \( K_n \) being closed, so is \( K \). Moreover \( K \subseteq [0, 1] \), so it is compact.

To prove that \( K \) is uncountable, consider the following bijection \( \varphi : K \to \{0, 1\}^\mathbb{N} \):
if \( x \in K \), then \( x \) is either in the left or right interval from \( K_1 \), and define \( \varphi(x)_0 = 0 \) if \( x \) is in the left interval, 1 otherwise. Iterations on \( K_2 \), etc defines \( \varphi(x) \), and \( \varphi \) is easily shown to be a bijection.

To prove that \( K \) has empty interior, assume \( x, y \in I \) for some interval \( I \subset K \). Then for any \( n \) we have \( x, y \) in the same interval from \( K_n \), i.e. \( \varphi(x)_n = \varphi(y)_n \). As \( \varphi \) is a bijection this imposes \( x = y \), so I needs to be a point, i.e. \( K \) has empty interior.

Finally, for any \( x \in K \), the set \( \{ y \in K : \forall k \leq n, \varphi(y)_k = \varphi(x)_k \} \) is infinite and its points are at distance at most \( 2^{-(n+1)} \) from \( x \), so \( x \) is an accumulation point.

(ii) An easy induction shows that the Lebesgue measure of \( K_n \) is \( (1 - d_0) \ldots (1 - d_{n-1}) \). So
\[
\text{Leb}(K) = \lim_{n \to \infty} (1 - d_0) \ldots (1 - d_{n-1}).
\]
This is 0 if the series of \( d_n \)'s diverges, a number in \( (0, 1) \) otherwise (for this analysis, take the logarithm and Taylor-expand).

Exercise 6. Let \( X \) be a nonnegative random variable. Prove that \( E(X) < +\infty \) if and only if \( \sum_{n \in \mathbb{N}} P(X \geq n) < \infty \).

Solution. By monotone convergence we have \( E(X) = \sum_{n \geq 0} P(X \in [n, n+1)) \), so that
\[
-1 + \sum_{n \geq 0} (n+1)P(X \in [n, n+1)) = \sum_{n \geq 0} nP(X \in [n, n+1)) \leq E(X) \leq \sum_{n \geq 0} (n+1)P(X \in [n, n+1)).
\]
The result follows by noting that \( \sum_{n \geq 0} (n+1)P(X \in [n, n+1)) = \sum_{n \in \mathbb{N}} P(X \geq n) \).

Exercise 7. Convergence in measure. Let \( (\Omega, \mathcal{A}, \mu) \) be a probability space, and \( (f_n)_{n \geq 1}, f : \Omega \to \mathbb{R} \) measurable (for the Borel \( \sigma \)-field on \( \mathbb{R} \)). We say that \( (f_n)_{n \geq 1} \) converges in measure to \( f \) if for any \( \varepsilon > 0 \) we have
\[
\mu(|f_n - f| > \varepsilon) \xrightarrow{n \to \infty} 0.
\]

(i) Show that \( \int |f - f_n| \, d\mu \to 0 \) implies that \( f_n \) converges to \( f \) in measure. Is the reciprocal true?
(ii) Show that if \( f_n \to f \ \mu\)-almost surely, then \( f_n \to f \) in measure. Is the reciprocal true?
(iii) Show that if \( f_n \to f \) in measure, there exists a subsequence of \( (f_n)_{n \geq 1} \) which converges \( \mu \)-almost surely.
(iv) (A stronger dominated convergence theorem) We assume that \( f_n \to f \) in measure and \( |f_n| \leq g \) for some integrable \( g : \Omega \to \mathbb{R} \), for any \( n \geq 1 \).
(a) Show that \( |f| \leq g \ \mu\)-a.s.
(b) Deduce that \( \int |f_n - f| \, d\mu \to 0 \).

Solution.
We first prove that
\[ f : \Omega \to \mathbb{R}, \quad \text{such that} \quad f_n \to f - a.s., \] so that
\[ f_n \text{ is included in the above set, this concludes the proof.} \]

(i) For any \( \varepsilon > 0 \) we have \( 1_{|f-f_n|>\varepsilon} \leq \frac{|f-f_n|}{\varepsilon} \), so that
\[ 
\mu(|f-f_n| > \varepsilon) \leq \frac{1}{\varepsilon} \int_{\Omega} |f_n - f| \, d\mu \to 0.
\]

The reciprocal is wrong, as shown by the example \( (f_n)_{n \geq 1} \) defined on \( ([0,1], \mathcal{B}, \text{Leb}) \) by \( f_n = n1_{[0,1/n]} \).

(ii) For any \( \varepsilon > 0 \) we have
\[ \cap_{n \geq 1} \cup_{m \geq n} \{|f_n - f| > \varepsilon\} \subset \{f_n \to f\} \]
so that \( \mu(\cap_{n \geq 1} \cup_{m \geq n} \{|f_m - f| > \varepsilon\}) = 0 \). The sequence \( (\cup_{m \geq n} \{|f_m - f| > \varepsilon\})_{n \geq 1} \)
decreases, so this implies that
\[ \lim_{n \to \infty} \mu(\cup_{m \geq n} \{|f_m - f| > \varepsilon\}) = 0, \]
and in particular \( \lim_{n \to \infty} \mu(\{|f_n - f| > \varepsilon\}) = 0 \).

The reciprocal is wrong, as shown by the example \( (f_{n,k})_{n \geq 1, 1 \leq k \leq n} \) defined on \( ([0,1], \mathcal{B}, \text{Leb}) \) by \( f_{n,k} = 1_{(k-1)/n,k/n]} \).

(iii) From the hypothesis, for any \( k \geq 1 \) there exists an index \( n_k \) such that
\[ \mu(|f_{n_k} - f| > 1/k) \leq 2^{-k}. \] Summability in \( k \) easily implies
\[ \mu(\cap_{m \geq 1} \cup_{k \geq m} \{|f_{n_k} - f| > 1/k\}) = 0. \]

As \( f_{n_k} \to f \) is included in the above set, this concludes the proof.

(iv) (a) For any \( \varepsilon > 0 \) we have
\[ \mu(|f| > g + \varepsilon) \leq \mu(|f| > |f_n| + \varepsilon) + \mu(|f - f_n| > \varepsilon) \]
so that \( \mu(|f| > g + \varepsilon) = 0 \). Therefore \( \mu \)-a.s. for any \( n \geq 1 \) we have \( |f| \leq g + 1/n \), so \( |f| \leq g \).

(b) We have
\[ \int_{\Omega} |f_n - f| \, d\mu = \int_{|f_n - f|<\varepsilon} |f_n - f| \, d\mu + \int_{|f_n - f|\geq\varepsilon} |f_n - f| \, d\mu \leq \varepsilon + 2 \int_{|f_n - f|\geq\varepsilon} |g| \, d\mu. \]

As \( g \) is integrable, \( \int_{|f_n - f|\geq\varepsilon} |g| \, d\mu \to 0 \) as \( n \to \infty \) (e.g. by dominated convergence or uniform continuity of the integral), and the conclusion follows as \( \varepsilon \) is arbitrary.

**Exercise 8.** Consider a probability space \( (\Omega, \mathcal{A}, \mu) \) and \( (A_n)_{n} \) a sequence in \( \mathcal{A} \). Let \( f : \Omega \to \mathbb{R} \) be measurable (for the Borel \( \sigma \)-field on \( \mathbb{R} \)) such that \( \int_{\Omega} |1_{A_n} - f| \, d\mu \to 0 \) as \( n \to \infty \). Prove that there exists \( A \in \mathcal{A} \) such that \( f = 1_A \mu \)-a.s., i.e. \( \mu(f = 1_A) = 1 \).

**Solution.** We first prove that \( |f| < 2 \mu \)-a.s.: \( \{|f| > 2\} \subset \{|f - 1_{A_n}| > 1\} \), so that
\[ \mu(|f| > 2) \leq \mu(|f - 1_{A_n}| > 1) \leq \int_{\Omega} |1_{A_n} - f| \, d\mu \to 0, \]
where for the second inequality we used \( \mu(X > 1) \leq \mathbb{E}(X) \) for any positive random variable. This proves \( |f| < 2 \mu \)-a.s.
We now prove that $f = f^2$ $\mu$-a.s., so that the expected result follows from the choice $A = \{f = 1\}$. We have

$$\int_{\Omega} |f - f^2| d\mu \leq \int_{\Omega} |f - 1_{A_n}| d\mu + \int_{\Omega} |f^2 - 1_{A_n}| d\mu$$

$$= \int_{\Omega} |f - 1_{A_n}| d\mu + \int_{\Omega} |f - 1_{A_n}| \cdot |f + 1_{A_n}| d\mu \leq 4 \int_{\Omega} 1_{A_n} - f| d\mu \to 0.$$  

where we used that $\mu$-a.s. $|f + 1_{A_n}| \leq |f| + 1 \leq 3$. This concludes the proof.

**Exercise 9.** Consider a probability space $(E, A, \mu)$ and $f_n : E \to \mathbb{R}$ measurable, $n \geq 1$. Assume $f_n \to f$ $\mu$-almost surely. Prove that for any $\varepsilon > 0$ there exists a set $A \in A$ such that $\mu(A) < \varepsilon$ and the convergence $f_n \to f$ is uniform on $A^c$.

**Solution.** Let

$$E_{k,n} = \cap_{j \geq n} \{|f_j - f| \leq 2^{-k}\}.$$  

As $f_j \to f$ a.s., $(E_{k,n})_{n \geq 1}$ is an increasing sequence converging to $E - N$ for some measurable $N$ with $\mu(N) = 0$. For any $k$ let $n_k$ be such that $\mu(E_{k,n_k}) \geq 1 - \frac{\varepsilon}{2^k}$, and let

$$A = N \cup \bigcup_{k \geq 1} (E - E_{k,n_k}).$$

Then $\mu(A) \leq \varepsilon$ and $f_n$ converges to $f$ uniformly on $A^c$: for any $\omega \in A^c$ we have $x \in E_{k,n_k}$ so that $|f_n(x) - f(x)| \leq 2^{-k}$ for all $n \geq n_k$. 
