Exercise 1. Let $n$ and $m$ be random numbers chosen independently and uniformly on $[1, N]$. What are $\Omega, \mathcal{A}$ and $\mathbb{P}$ (which all implicitly depend on $N$)? Prove that $\mathbb{P}(n \land m = 1) \xrightarrow{N \to \infty} \zeta(2)^{-1}$ where $\zeta(2) = \prod_{p \in \mathbb{P}}(1 - p^{-2})^{-1} = \sum_{n \geq 1} n^{-2} = \frac{\pi^2}{6}$ (you don’t have to prove these equalities). Here $\mathbb{P}$ is the set of prime numbers and $n \land m = 1$ means that their greatest common divisor is 1.

Solution. We can choose $\Omega = [1, N]^2$, $\mathcal{A} = \{A : A \subset \Omega\}$ and define $\mathbb{P}$ through $\mathbb{P}(A) = \frac{|A|}{N^2}$. We denote the random variable $(n, m) = (n(\omega), m(\omega))$. Let $A_p = \{p \mid n$ and $p \mid m\}$ and $f_N = \{p \leq N, p \in \mathbb{P}\}$. Then, by inclusion-exclusion,

$$\mathbb{P}(n \land m = 1) = \mathbb{P}(\cap_{p \leq N} A_p) = 1 - \mathbb{P}(\cup_{p \leq N} A_p) = 1 - \sum_{p_1 \leq N} \mathbb{P}(A_{p_1}) + \sum_{p_1 < p_2 \leq N} \mathbb{P}(A_{p_1} \cap A_{p_2}) + \cdots + (-1)^f \mathbb{P}(A_{p_1} \cap A_{p_2} \cap \cdots \cap A_{p_{f_N}})$$

$$= 1 - \sum_{p_1 \leq N} \left(\frac{|N/p_1|}{N}\right) + \sum_{p_1 < p_2 \leq N} \left(\frac{|N/(p_1 p_2)|}{N}\right) - \cdots$$

When replacing the integral parts by fractions, we obtain

$$1 - \sum_{p_1 \leq N} p_1^{-2} + \sum_{p_1 < p_2 \leq N} (p_1 p_2)^{-2} - \cdots + (-1)^f (p_1 \cdots p_{f_N})^{-2} = (1 - p_1^{-2}) \cdots (1 - p_{f_N}^{-2})^2,$$

which converges to the expected result as $N \to \infty$. Hence we now just need to bound the error when replacing integral parts with fractions. Top bound this error we use $|a^2 - |a|^2| \leq (a - \bar{a}) \cdot 2a$ for any $a > 0$ so that

$$\left| \left(\frac{|N/(p_1 p_2 \cdots p_k)|}{N}\right)^2 - \left(\frac{(N/(p_1 p_2 \cdots p_k))}{N}\right)^2 \right| \leq \frac{2}{N p_1 p_2 \cdots p_k}$$

and the global error is at most

$$N^{-1} \left( \sum_{p_1 \leq N} p_1^{-1} + \sum_{p_1 < p_2 \leq N} (p_1 p_2)^{-1} + \cdots + (p_1 \cdots p_{f_N})^{-1} \right) = N^{-1} (1 + p_1^{-1}) \cdots (1 + p_{f_N}^{-1}) \to 0,$$

where the last convergence relies on $\log[(1 + p_1^{-1}) \cdots (1 + p_{f_N}^{-1})] \sim \sum_{p \leq N} 1/p \sim \log \log N$.

Exercise 2. Let $X$ be a random variable with density $f_X(x) = (1 - |x|) \mathbb{1}_{(-1,1)}(x)$. Show that its characteristic function is

$$\phi_X(u) = \frac{2(1 - \cos u)}{u^2}.$$

Solution. Let’s calculate:

$$\frac{1}{2} \int_{\mathbb{R}} e^{iux} f_X(x)dx = \Re \int_{0}^{1} (1 - x)e^{iux} dx$$

$$= \Re \int_{0}^{1} e^{iux}dx - \Re \frac{e^{iu}}{iu} + \Re \int_{0}^{1} \frac{e^{iux}}{iu} dx$$

$$= \Re \frac{e^{iu} - 1}{iu} - \Re \frac{e^{iu}}{iu} + \Re \frac{e^{iu} - 1}{(iu)^2}$$

$$= - \Re \frac{e^{iu} - 1}{u^2},$$
where we used integration by parts in the second equality. This concludes the proof.

**Exercise 3.**

1. Prove that \( \hat{\mu} \) is real-valued if and only if \( \mu \) is symmetric, i.e. \( \mu(A) = \mu(-A) \) for any Borel set \( A \).

2. If \( X \) and \( Y \) are i.i.d., prove that \( X - Y \) has a symmetric distribution.

**Solution.**

1. If \( \mu(A) = \mu(-A) \) for any Borel set \( A \), then
\[
\hat{\mu}(u) = \int_{-\infty}^{\infty} e^{-ixu} \mu(dx) = \int_{-\infty}^{\infty} e^{ixu} \mu(-dx) = \int_{-\infty}^{\infty} e^{ixu} \mu(dx) = \hat{\mu}(u),
\]
so \( \mu \) is real-valued.

Conversely, from the previous calculation, if \( \hat{\mu} \) is real-valued, then for any \( u \in \mathbb{R} \) we have
\[
\int_{-\infty}^{\infty} e^{ixu} (\mu(dx) - \mu(-dx)) = 0.
\]
By characterization through Fourier transform, this implies that \( \mu(dx) - \mu(-dx) = 0 \) (as a signed measure), hence the claim.

2. From the previous question we just need to prove that the characteristic function is real-valued, which is true:
\[
E(e^{iu(X-Y)}) = E(e^{iuX})E(e^{-iuY}) = E(e^{iuX})E(e^{-iuX}) = E(e^{iuX})E(e^{-iuX}) \in \mathbb{R}.
\]

**Exercise 4.** Let \( X_\lambda \) be a real random variable, with Poisson distribution with parameter \( \lambda \). Calculate the characteristic function of \( X_\lambda \). Conclude that \( (X_\lambda - \lambda) / \sqrt{\lambda} \) converges in distribution, as \( \lambda \to \infty \).

**Solution.** Let’s simply calculate:
\[
E(e^{iuX_\lambda}) = \sum_{k \geq 0} \frac{\lambda^k}{k!} e^{-\lambda} e^{iku} = e^{\lambda(e^{iu} - 1)}.
\]
The desired weak convergence now follows from the choice \( u = v / \sqrt{\lambda} \) and Taylor expansion in \( \lambda \) for fixed \( v \).

**Exercise 5.** Assume that the sequence of random variables \( (X_n)_{n \geq 1} \) satisfies \( E X_n \to 1 \) and \( E X_n^2 \to 1 \). Prove that \( (X_n)_{n \geq 1} \) converges in distribution. What is the limit?

**Solution.** We will prove that for smooth compactly supported \( f \) we have \( E f(X_n) \to E f(1) \), which is sufficient to characterize the weak convergence of \( X_n \) to 1. We have \( |f(X_n) - f(1)| \leq \|f'\|_{\infty} \cdot |X_n - 1| \), so that by taking expectation first and using the Cauchy-Schwarz inequality after we obtain
\[
E |f(X_n) - f(1)| \leq \|f'\|_{\infty} \cdot E |X_n - 1| \leq \|f'\|_{\infty} \cdot (E[|X_n - 1|^2])^{1/2}.
\]
By expansion, and using our assumptions, we have \( E[|X_n - 1|^2] \to 0 \), which concludes the proof.
Exercise 6. Let \((X_n)_{n \geq 1}, (Y_n)_{n \geq 1}\) be real random variables, with \(X_n\) and \(Y_n\) independent for any \(n \geq 1\), and assume that \(X_n\) converges in distribution to \(X\) and \(Y_n\) to \(Y\). Prove that \(X_n + Y_n\) converges in distribution to \(X + Y\).

Solution. The proof is easy with characteristic functions, using successively independence and weak convergence: for any \(\lambda \in \mathbb{R}\) we have
\[
\mathbb{E}(e^{i\lambda (X_n + Y_n)}) = \mathbb{E}(e^{i\lambda X_n}) \mathbb{E}(e^{i\lambda Y_n}) \xrightarrow{n \to \infty} \mathbb{E}(e^{i\lambda X}) \mathbb{E}(e^{i\lambda Y}) = \mathbb{E}(e^{i\lambda (X+Y)}).
\]

Exercise 7. Let \(X, Y\) be independent and assume that for some constant \(\alpha\) we have \(\mathbb{P}(X + Y = \alpha) = 1\). Prove that \(X\) and \(Y\) are both constant random variables.

Solution. Assume the contrary, i.e. e.g. the exists \([a, b]\) and \([c, d]\) with \(b < c\) such that \(\mathbb{P}(X \in [a, b]) > 0, \mathbb{P}(X \in [c, d]) > 0\).

Let \([x, y]\) be such that \(y - x < c - b\) and \(\mathbb{P}(Y \in [x, y]) > 0\).

Then we have
\[
\mathbb{P}(X + Y > c + x) \geq \mathbb{P}(X > c, Y > x) > 0
\]
and
\[
\mathbb{P}(X + Y < b + y) \geq \mathbb{P}(X < b, Y < y) > 0.
\]

The two equations above together with \(c + x > b + y\) contradict the fact that \(X + Y\) has distribution a Dirac mass.

Exercise 8. Let \(f, g : \mathbb{R} \to \mathbb{R}\) be nondecreasing measurable functions. Let \(\mu\) be a probability measure on \(\mathbb{R}\) and assume \(f, g, fg \in L^1(\mu)\). prove that
\[
\int fg d\mu \geq \int f d\mu \cdot \int g d\mu.
\]

Solution. We have \(\iint (f(x) - f(y)) \cdot (g(x) - g(y)) d\mu(x) d\mu(y) \geq 0\). Then the result follows from an expansion and Fubini’s theorem.

Exercise 9. Let \((X_n)_{n \geq 1}\) be a sequence of i.i.d. random variables with standard Cauchy distribution and let \(M_n = \max(X_1, \ldots, X_n)\). Prove that \((nM_n^{-1})_{n \geq 1}\) converges in distribution and identify the limit.

Solution. For any fixed \(x > 0\) we have
\[
\mathbb{P}(nM_n^{-1} > x) = \mathbb{P}(M_n < n/x)
= \prod_{k=1}^{n} \mathbb{P}(X_k < n/x)
= \left(1 - \int_{n/x}^{\infty} \frac{du}{\pi(1 + u^2)}\right)^n
= \exp\left(n \log(1 - \frac{1}{\pi n/x} + O(1/n^2))\right)
\to \exp(-x/\pi)
= \int_{x}^{\infty} \pi^{-1} e^{-u/\pi} du,
\]
where we used independence for the second equality. Therefore $nM^{-1}$ converges in distribution to an exponential random variable with parameter $\pi^{-1}$.

**Exercise 10.** Let $(X_i)_{i \geq 1}$ be a sequence of independent random variables, with $X_i$ uniform on $[-i, i]$. Let $S_n = X_1 + \cdots + X_n$. Prove that $S_n/n^{3/2}$ converges in distribution and describe the limit.

**Solution.** Let’s calculate the characteristic function ($Y_k$’s denote independent uniform random variables on $[-1, 1]$):

$$
\mathbb{E}(\exp(iuS_n n^{-3/2})) = \prod_{k=1}^{n} \mathbb{E}(\exp(iu n^{-3/2} X_k)) = \prod_{k=1}^{n} \mathbb{E}(\exp(iu n^{-3/2} Y_k) = \prod_{k=1}^{n} \frac{\sin(un^{-3/2}k)}{un^{-3/2}}
$$

As $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6 \sim n^3/3$, we obtain

$$
\mathbb{E}(\exp(iuS_n n^{-3/2})) \to e^{-\frac{u^2}{2n^2}},
$$

in other words $S_n n^{-3/2}$ converges in distribution to a Gaussian random variable with mean zero and variance $3^2$.

**Exercise 11.** Find a probability distribution $\mu$ of a $\mathbb{Z}$-valued random variable which is symmetric ($\mu\{i\} = \mu\{-i\}$ for any $i \in \mathbb{Z}$), not integrable, but such that its characteristic function is differentiable at 0.

**Solution.** One example is $\mu\{n\} = \mu\{-n\} = \frac{c}{n^2 \log n}$, $n \geq 2$, 0 otherwise, where $c$ is a normalizing constant. Clearly $\int |x| d\mu(x) = \infty$, but

$$
\hat{\mu}(u) = c \sum_{k \geq 2} \frac{\cos(ku)}{k^2 \log k}
$$

is differentiable at 0 because

$$
\frac{\hat{\mu}(\varepsilon) - \hat{\mu}(0)}{\varepsilon} = \frac{c'}{\varepsilon} \sum_{k=2}^{(\log(\varepsilon)^{-1})^{-1}} \frac{1 - \cos(k\varepsilon)}{k^2 \log k} + O \left( \frac{1}{\sqrt{\log \varepsilon}^{-1}} \right)
$$

From $|1 - \cos x| \leq Cx^2$ one gets the result that $\hat{\mu}$ is differentiable at 0, with differential 0.

**Exercise 12** Let $X, Y$ be i.i.d., with characteristic functions denoted $\varphi_X, \varphi_Y$, and suppose $\mathbb{E}(X) = 0, \mathbb{E}(X^2) = 1$. Assume also that $X+Y$ and $X-Y$ are independent.

1. Prove that

$$
\varphi_X(2u) = (\varphi_X(u))^3 \varphi_X(-u)
$$

2. Prove that $X$ is a standard Gaussian random variable.

**Solution.**
(1) Let’s calculate:

$$\varphi_X(2u) = \mathbb{E}(e^{iu(X+Y+X-Y)}) = \mathbb{E}(e^{iuX}) \mathbb{E}(e^{iuY})$$

$$= \mathbb{E}(e^{iuX}) \mathbb{E}(e^{iuY}) \mathbb{E}(e^{-iuY}) = (\mathbb{E}(e^{iuX}))^3 \mathbb{E}(e^{-iuX}),$$

which concludes the proof of the first question.

(2) This is a truly hard question. Let $\varphi = \varphi_X$. By induction on $n$, one first obtains (from the previous question applied to $u$ and $-u$) that

$$\frac{\varphi(u)}{\varphi(-u)} = \left( \frac{\varphi(u2^{-n})}{\varphi(-u2^{-n})} \right)^2$$

for any $n \geq 0$. This actually holds only for $|u| < \varepsilon$ where $\varepsilon$ is small enough so that none denominator above vanishes (possible by dominated convergence, think about it!).

Now let $n \to \infty$ in the previous equation. As $\mathbb{E}(X) = 0$, and $\mathbb{E}(X^2) < \infty$, a Taylor expansion of the exponential shows that the right hand side converges to $1$.

We have therefore proved that $\varphi(u) = \varphi(-u)$ for $|u| < \varepsilon$. This information injected in the result of Section 1 gives $\varphi(2u) = \varphi(u)^4$. Now an iteration of this relation provides

$$\varphi(u) = \varphi(u/2^n)^{4^n}.$$ 

By Taylor expansion again, the above right hand side converges, as $n \to \infty$, to $e^{-u^2/2}$, which concludes the proof.