Probability, homework 5, due October 11.

As a preliminary to this homework, for exercises 3 and 4 read the Borel-Cantelli lemma (Lemma 3.4 in Varadhan’s Probability Theory book).

Exercise 1. Prove that if a sequence of real random variables \((X_n)\) converge in distribution to \(X\), and \((Y_n)\) converges in distribution to a constant \(c\), then \(X_n + Y_n\) converges in distribution to \(X + c\).

Exercise 2. Assume that \((X, Y)\) has joint density
\[ c e^{-(1+x^2)(1+y^2)}, \]
where \(c\) is properly chosen. Are \(X\) and \(Y\) Gaussian random variables? Is \((X, Y)\) a Gaussian vector?

Exercise 3. Let \(\epsilon > 0\) and \(X\) be uniformly distributed on \([0, 1]\). Prove that, almost surely (i.e. the following event has probability 1), there exists only a finite number of rationals \(\frac{p}{q}\), with \(p \land q = 1\), such that
\[ |X - \frac{p}{q}| < \frac{1}{q^2 + \epsilon}. \]

Exercise 4. You toss a coin repeatedly and independently. The probability to get a head is \(p\), a tail is \(1-p\). Let \(A_k\) be the following event: \(k\) or more consecutive heads occur amongst the tosses numbered \(2^k, \ldots, 2^{k+1} - 1\). Prove that \(P(A_k \text{ i.o.}) = 1\) if \(p \geq 1/2\), 0 otherwise.

Here, i.o. stands for “infinitely often”, and \(A_k\) i.o. is the event \(\cap_{n \geq 1} \cup_{m \geq n} A_m\).

Exercise 5. Prove that for any \(x > 0\), \(\frac{1}{x} = \int e^{-tx} dt\). Deduce the value of \(\int_0^\infty \frac{\sin x}{x} dx\).

Exercise 6. For any probability measure \(\mu\) supported on \(\mathbb{R}_+\), one defines the Laplace transform as
\[ \mathcal{L}_\mu(\lambda) = \int_0^\infty e^{-\lambda x} d\mu(x), \ \lambda \geq 0. \]

(1) Prove that \(\mathcal{L}_\mu\) is well-defined, continuous on \(\mathbb{R}_+\) and \(C^\infty\) on \(\mathbb{R}_+^*\).
(2) Prove that \(\mathcal{L}_\mu\) characterizes the probability measure \(\mu\) supported on \(\mathbb{R}_+\).
(3) Assume that for a sequence \((\mu_n)_{n \geq 1}\) of probability measure supported on \(\mathbb{R}_+\), one has \(\mathcal{L}_{\mu_n}(\lambda) \rightarrow \ell(\lambda)\) for any \(\lambda \geq 0\), and \(\ell\) is right-continuous at 0. Prove that \((\mu_n)_{n \geq 1}\) is tight, and that it converges weakly to a measure \(\mu\) such that \(\ell = \mathcal{L}_\mu\).