Exercise 1. Let \((X_i)_{i \geq 1}\) be i.i.d. Gaussian with mean 1 and variance 3. Show that
\[
\lim_{n \to \infty} \frac{X_1 + \cdots + X_n}{n} = \frac{1}{4} \text{ a.s.}
\]

Solution. Let \(A = \{\lim_{n \to \infty} \frac{X_1 + \cdots + X_n}{n} = 1\}\) and \(B = \{\lim_{n \to \infty} \frac{X_1^2 + \cdots + X_n^2}{n} = 4\}\). The \(X_i\)'s and \(X_i^2\)'s are in \(L^1\) and \(\mathbb{E}(X_i) = 1, \mathbb{E}(X_i^2) = 3 + \mathbb{E}(X_i)^2 = 4\), so by the law of large numbers \(\mathbb{P}(A) = \mathbb{P}(B) = 1\), so that \(\mathbb{P}(A \cap B) = 1\). As \(A \cap B \subset \{\lim_{n \to \infty} \frac{X_1 + \cdots + X_n}{n} = \frac{1}{4}\}\), the result follows.

Exercise 2. Let \(f\) be a continuous function on \([0, 1]\). Calculate the asymptotics, as \(n \to \infty\), of
\[
\int_{[0,1]^n} f \left( \frac{x_1 + \cdots + x_n}{n} \right) \, dx_1 \cdots dx_n.
\]

Solution. Let \(U_1, U_2, \ldots\) be independent random variables, uniform on \([0, 1]\). By the law of large numbers, \((U_1 + \cdots + U_n)/n \to 1/2\) almost surely, hence in distribution. This implies that for any continuous function on \([0, 1]\) (and therefore bounded),
\[
\mathbb{E} f \left( \frac{U_1 + \cdots + U_n}{n} \right) \to f \left( \frac{1}{2} \right),
\]
which is equivalent to showing that the desired asymptotics is \(f \left( \frac{1}{2} \right)\).

Exercise 3. The goal of this exercise is to prove that any function, continuous on an interval of \(\mathbb{R}\), can be approximated by polynomials, arbitrarily close for the \(L^\infty\) norm (this is the Bernstein-Weierstrass theorem). Let \(f\) be a continuous function on \([0, 1]\). The \(n\)-th Bernstein polynomial is
\[
B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f \left( \frac{k}{n} \right).
\]

- a) Let \(S_n(x) = B^{(n,x)}/n\), where \(B^{(n,x)}\) is a binomial random variable with parameters \(n\) and \(x\): \(B^{(n,x)} = \sum_{i=1}^{n} X_i\) where the \(X_i\)'s are independent and \(\mathbb{P}(X_i = 1) = x, \mathbb{P}(X_i = 0) = 1 - x\). Prove that \(B_n(x) = \mathbb{E}(f(S_n(x)))\).
- b) Prove that \(\|B_n - f\|_{L^\infty([0,1])} \to 0\) as \(n \to \infty\).

Solution.

(a) We have \(\mathbb{P}(B^{(n,x)} = k) = \binom{n}{k} x^k (1-x)^{n-k}\), so that
\[
\mathbb{E}(f(S_n(x))) = \sum_{k=0}^{n} \mathbb{P}(B^{(n,x)} = k) \cdot f \left( \frac{k}{n} \right) = B_n(x).
\]

(b) Let \(\varepsilon > 0\). As \(f\) is continuous on the compact \([0, 1]\), it is uniformly continuous, i.e. there exists \(\delta > 0\) such that \(|f(a) - f(b)| < \varepsilon\) for any \(a, b \in [0,1]\) satisfying \(|a - b| < \delta\). Then
\[
|B_n(x) - f(x)| = |\mathbb{E}(f(S_n(x)) - f(x))| \leq |\mathbb{E}(|f(S_n(x)) - f(x)||S_n(x) - x| < \delta)| + |\mathbb{E}(|f(S_n(x)) - f(x)||S_n(x) - f(x))|_{|S_n(x) - f(x)| \geq \delta}|.
\]
\[
\leq \varepsilon + 2\|f\| \mathbb{P}(|S_n(x) - x| > \delta) \leq \varepsilon + 2\|f\| \delta^{-2} \mathbb{E}([S_n(x) - x]^2).
\]
We have \(\mathbb{E}([S_n(x) - x]^2) = n^{-2} \text{Var}(\sum X_i) = n^{-2} \sum \text{Var}(X_i) = n^{-1} \text{Var}(X_1)\), so that for \(n > 2\|f\| \delta^{-2} \max_{0 \leq x \leq 1} \text{Var}(X_1)\) we obtain \(|B_n(x) - f(x)| \leq 2\varepsilon\), which concludes the proof.
Exercise 4. Calculate
\[ \lim_{n \to \infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}. \]

Solution. Let \( X_n \) be a Poisson random variable with parameter \( n \). Then the sum of interest can exactly be interpreted as
\[ P(X_n \leq n) = P\left( \frac{X_n - n}{\sqrt{n}} \leq 0 \right). \]

Then, as \( n \to \infty \), to \( P(\mathcal{N}(0,1) \leq 0) = \frac{1}{2} \) from the central limit theorem for \( X_n \) (which can be proved directly by characteristic functions).

Exercise 5. Let \( (X_n)_{n \geq 0} \) be a sequence of i.i.d random variables, with uniform distribution on \([0,1]\). Let \( Y_n = (X_n)^n \).

1. Calculate the distribution of \( Y_n \).
2. Show that \( (Y_n)_{n \geq 0} \) converges to 0 in probability.
3. Show that \( (Y_n)_{n \geq 0} \) converges in \( L^1 \).
4. Show that almost surely \( (Y_n)_{n \geq 0} \) does not converge.

Solution. (1) We just calculate the cumulative distribution function: For any \( x \in [0,1] \) we have \( P(Y_n \leq x) = P(X_n \leq x^{1/n}) = x^{1/n} \).

(2) From the previous calculation, \( P(|Y_n| > \varepsilon) = 1 - \varepsilon^{1/n} \to 0 \) as \( n \to \infty \), so \( Y_n \) converges in probability to 0.

(3) We have \( E[|Y_n|] = \int_0^1 P(Y_n > x) \), and from the previous question the integrand converges pointwise to 0. So by dominated convergence \( E[|Y_n|] \to 0 \) so \( Y_n \to 0 \) in \( L^1 \).

(4) As \( Y_n \) converges to 0 in probability, if it converges a.s. then it can only be to 0. But \( \sum_{n \geq 1} P(|Y_n| > 1/2) \) diverges from the calculation in (2), and the \( Y_n \)'s independent, so that a.s. \( Y_n > 1/2 \) infinitely often. Hence it does not converge a.s.

Long problem. The goal is to prove the Erdös-Kac theorem: if \( w(m) \) denotes the number of distinct prime factors of \( m \) and \( k \) is a random variable uniformly distributed on \([1,n]\), then the following convergence in distribution holds:
\[ \frac{w(k) - \log \log n}{\sqrt{\log \log n}} \to \mathcal{N}(0,1). \]

1. Prove that if \( (X_n)_{n \geq 1} \) converges in distribution to \( \mathcal{N}(0,1) \) and \( \sup_{n \geq 1} E[X_n^2] < \infty \) for any \( k \in \mathbb{N} \), then \( \lim_{n \to \infty} E[X_n^k] = E[\mathcal{N}(0,1)^k] \) for any \( k \in \mathbb{N} \).

2. Prove that for any \( x \in \mathbb{R} \) and \( d \geq 1 \) we have
\[ \left| e^{ix} - \sum_{\ell=0}^{d} \frac{(ix)^\ell}{\ell!} \right| \leq \frac{|x|^{d+1}}{(d+1)!}. \]

3. Assume that \( \lim_{n \to \infty} E[X_n^k] = E[\mathcal{N}(0,1)^k] \) for any \( k \in \mathbb{N} \). Prove that \( X_n \) converges in distribution to \( X \).

4. Let \( w_y(m) \) be the number of prime factors of \( m \) which are smaller than \( y \). Let \( (B_p)_{p \text{ prime}} \) be independent random variables such that \( P(B_p = 1) = 1 - P(B_p = 0) = \frac{1}{p} \), \( W_y = \sum_{p \leq y} B_p \), \( \mu_y = \sum_{p \leq y} \frac{1}{p} \), \( \sigma_y^2 = \sum_{p \leq y} \frac{1}{p^2} - \frac{1}{p^2} \).
Prove that if $y = n^{o(1)}$, then for any $d \in \mathbb{N}$ we have
\[
\mathbb{E}
\left[
\frac{w_y(k) - \mu_y}{\sigma_y}
\right]^d
\rightarrow 0
\]
as $n \rightarrow \infty$.

**Solution.**

(i) For $M > 0$, let $f_M(x) = x^k$ if $|x| < M$, $M$ otherwise. Then
\[
|\mathbb{E}[X_n^k - \mathcal{N}(0,1)^k]|
\leq|\mathbb{E}[f_M(X_n) - f_M(\mathcal{N}(0,1))]| + |\mathbb{E}[|\mathcal{N}(0,1)^k||X_n| > M]| + |\mathbb{E}[\mathcal{N}(0,1)^k||X_n| > M]|
\leq|\mathbb{E}[f_M(X_n) - f_M(\mathcal{N}(0,1))]| + |\mathbb{E}[X_n^2k/M^k]| + |\mathbb{E}[\mathcal{N}(0,1)^{2k}/M^k]|
\leq|\mathbb{E}[f_M(X_n) - f_M(\mathcal{N}(0,1))]| + C_k/M^k.
\]

For fixed $\varepsilon > 0$, one can find $M > 0$ such that $C_k/M^k < \varepsilon$. For this $M$, by weak convergence the expectation above converges to 0. Hence for large enough $n$ we have $|\mathbb{E}[X_n^k - \mathcal{N}(0,1)^k]| \leq 2\varepsilon$.

(ii) This is an easy exercise from the Taylor formula with integral remainder.

(iii) Let $w \in \mathbb{R}$ be fixed. One easily proves that $\mathbb{E}[|\mathcal{N}(0,1)^{2k}|w^{2k}] \ll (2\ell)!$. Hence by the previous question for any $\varepsilon > 0$ we can find $d > 0$ such that
\[
|\mathbb{E}[e^{iwX_n} - e^{iw\mathcal{N}(0,1)}]| \leq \sum_{\ell=0}^d|\mathbb{E}[X_n^\ell - \mathcal{N}(0,1)^\ell]| + \varepsilon.
\]

Convergence of the moments then gives $|\mathbb{E}[e^{iwX_n} - e^{iw\mathcal{N}(0,1)}]| \leq 2\varepsilon$ for large enough $n$, which concludes the proof.

(iv) Note that, by binomial expansion,
\[
\mathbb{E}[(w_y(k) - \mu_y)^d] - \mathbb{E}[(W_y - \mu_y)^d] = \sum_{j=0}^d \binom{d}{j}(-\mu_y)^{d-j}(\mathbb{E}[(w_y(k))^j] - \mathbb{E}[(W_y)^j]),
\]
thus we only need to prove that for any fixed $j$,
\[
\mathbb{E}[(w_y(k))^j] - \mathbb{E}[(W_y)^j] = o(\mu_y^{-\frac{j}{2}}).
\]

We have
\[
\mathbb{E}[(w_y(k))^j] = \sum_{p_1, \ldots, p_j \leq y} \mathbb{P}(p_1 | k, \ldots, p_j | k) = \sum_{p_1, \ldots, p_j \leq y} \frac{|n/\text{lcm}(p_1, \ldots, p_j)|}{n},
\]
and
\[
\sum_{p_1, \ldots, p_j \leq y} \frac{1}{\text{lcm}(p_1, \ldots, p_j)} + O(y^j/n) = \mathbb{E}[(W_y)^j] + O(y^j/n) = \mathbb{E}[(W_y)^j] + o(\mu_y^{-\frac{j}{2}}),
\]
concluding the proof.

(v) We now choose $y = n^{1/\log \log \log n}$. From the previous questions, $\frac{w_y(k) - \mu_y}{\sigma_y}$ converges to a standard Gaussian. Any $k \leq n$ has at most $\log n/\log y = \log \log \log n$ prime factors in $[y, n]$, we have $w_y(k) = w(k) + O(\log \log \log n)$. Moreover a calculation gives $\mu_y = \log y + O(1) = \log n + O(\log \log \log n)$ and $\sigma_y^2 \sim \log \log n$, so that $\frac{w(k) - \mu_y}{\sigma_y}$ also converges in distribution to a standard Gaussian. Injecting the estimates on $\mu_y$ and $\sigma_y$ concludes the proof.