Probability, homework 6, due April 1st.

Exercise 1. Let $(Y_n)_{n\in\mathbb{N}^*}$ be a sequence of random variables, and assume (Y_n) converges in distribution to a limiting Y. Also, on some probability space $(\Omega,\mathcal{F},\mathbb{P})$, the sequence of independent random variables $X := (X_n)_{n\in\mathbb{N}^*}$ is defined, and we assume that the sequence of partial sums $(S_n)_{n\in\mathbb{N}}$ (i.e. $S_0 = 0$ and $S_n := \sum_{j=1}^n X_j$) converges in distribution. Set $(\mathcal{F}_n)$ the natural filtration of X and $\Phi_n(t) = \mathbb{E}(\exp(\mathrm{i}tS_n))$ for $t \in \mathbb{R}$.

(i) Establish that $(\Phi_{Y_n}(\cdot))_{n\geq 1}$ converges uniformly on every compact, i.e. show that for any $a > 0$, $\max_{t\in[-a,a]} |\Phi_{Y_n}(t) - \Phi_Y(t)| \to 0$ as $n \to \infty$. Establish moreover that there exists $a > 0$ such that for any $n \geq 1$, $\min_{t\in[-a,a]} |\Phi_{Y_n}(t)| \geq 1/2$.

(ii) Show that there exists $t_0 > 0$ such that if $t \in [-t_0, t_0]$, then $(\exp(\mathrm{i}tS_n)/\Phi_n(t))_{n\geq 0}$ is a $(\mathcal{F}_n)-$martingale (i.e. both its real and imaginary parts are martingales).

(iii) Prove that we can choose $t_0 > 0$ such that for any $t \in [-t_0, t_0]$, $\lim_{n\to\infty} \exp(\mathrm{i}tS_n)$ exists $\mathbb{P}$-a.s.

(iv) Set
$$C = \{(t,\omega) \in [-t_0, t_0] \times \Omega : \lim_{n\to\infty} \exp(\mathrm{i}tS_n(\omega)) \text{ exists}\}.$$
Prove that C is measurable, i.e. in the product of $\mathcal{B}([-t_0, t_0])$ with $\mathcal{F}$.

(v) Establish that $\int_{-t_0}^{t_0} \mathbb{1}_C(t,\omega)\mathbb{P}(\mathrm{d}\omega)\mathrm{d}t = 2t_0$.

(vi) Prove that $\lim_{n\to\infty} S_n$ exists $\mathbb{P}$-a.s.

Exercise 2. Let B be a Brownian motion.

(i) Calculate $\mathbb{E}\left(B_s B_t^2\right)$, $\mathbb{E}\left(B_t \mid \mathcal{F}_s\right)$, $\mathbb{E}\left(B_t \mid B_s\right)$, for $t > s > 0$.

(ii) What is $\mathbb{E}\left(B_s^2 B_t^2\right)$, still for $t > s$?

(iii) What is the law of $B_t + B_s$? Same question for $\lambda_1 B_{t_1} + \cdots + \lambda_k B_{t_k}$ ($0 < t_1 < \cdots < t_k$) ? What is the law of $\int_0^1 B_s \mathrm{d}s$?

Exercise 3. Let B be a Brownian motion.

(i) Study the convergence in probability of $\frac{\log(1+B_t^2)}{\log t}$ as $t \to \infty$.

(ii) What about the almost sure convergence of $\frac{\log(1+B_t^2)}{\log t}$ as $t \to \infty$?

Exercise 4. Let B be a Brownian motion, and for any $t \in [0,1]$ define
$$W_t = B_t - tB_1.$$
It is called a Brownian bridge.

(i) Prove that W is a Gaussian process and calculate its covariance.

(ii) Let $0 < t_1 < \cdots < t_k < 1$. Prove that the vector $(W_{t_1}, W_{t_2}, \ldots, W_{t_k})$ has dentity
$$f(x_1,\ldots,x_k) = \sqrt{2\pi}\, p_{t_1}(x_1) p_{t_2-t_1}(x_2 - x_1) \ldots p_{t_k - t_{k-1}}(x_k - x_{k-1}) p_{1-t_k}(x_k)$$
where $p_t(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

(iii) Prove that the law of $(W_{t_1}, W_{t_2}, \ldots, W_{t_k})$ is the same as the law of $(B_{t_1}, B_{t_2}, \ldots, B_{t_k})$ conditionally to $B_1 = 0$.

(iv) Prove that the processes $(W_t)_{0\leq t\leq 1}$ and $(W_{1-t})_{0\leq t\leq 1}$ have the same distribution.

Exercise 5. Let B be a Brownian motion, and for any $t \geq 0$ define

$$Z_t = B_t - \int_0^t \frac{B_s}{s} \mathrm{d}s.$$

Prove that Z is a Gaussian process and calculate its covariance. Does this process have a famous name?

Exercise 6. Martingales from Brownian motion. Amongst the following processes, which ones are $\mathcal{F}$-martingales, where $\mathcal{F}$ is the natural filtration of $(B_s, s \geq 0)$? $B_t^2 - t$, $B_t^3 - 3\int_0^t B_s \mathrm{d}s$, $B_t^3 - 3tB_t$, $tB_t - \int_0^t B_s \mathrm{d}s$.

Exercise 7. Scaling and equalities in law.

(i) Let $T_a = \inf\{t \mid B_t = a\}$ and $S_1 = \sup\{B_s, s \leq 1\}$. Prove that $T_a \overset{\text{law}}{=} a^2 T_1$. Prove that $T_1 \overset{\text{law}}{=} 1/S_1^2$.

(ii) Let $g_t = \sup\{s \leq t \mid B_s = 0\}$ and $d_t = \inf\{s \geq t \mid B_s = 0\}$. Prove that g is not a stopping time, and that d is a stopping time. Prove that $g_t \overset{\text{law}}{=} t g_1$, $d_t \overset{\text{law}}{=} t d_1$, $g_t \overset{\text{law}}{=} \frac{t}{d_1} \overset{\text{law}}{=} \frac{1}{d_{1/t}}$.

Exercise 8. A convergence in law. Prove that as $t \to \infty$, $\left(\int_0^t e^{B_s} \mathrm{d}s\right)^{1/\sqrt{t}}$ converges in law towards $e^{|\mathcal{N}|}$, where $\mathcal{N}$ is a standard Gaussian random variable.

Exercise 9. The zeros of Brownian motion. Let $\mathcal{Z} = \{t \geq 0 \mid B_t = 0\}$.

(i) Prove that $\mathcal{Z}$ is almost surely a closed and unbounded set, with no isolated points.

(ii) Prove that $\mathcal{Z}$ is almost surely uncountable.