Probability, homework 7, due April 29th.

Exercise 1. For $a > 0$, let $\sigma_a = \inf\{t \geq 0 : B_t < t - a\}$.

(i) Prove that σ_a is a.s. finite and that $\lim_{a\to\infty} \sigma_a = \infty$ almost surely.

(ii) Prove that $\mathbb{E}(e^{\frac{1}{2}\sigma_a}) = e^a$. For this, use the martingale $e^{-(\sqrt{1+2\lambda}-1)(B_t-t)-\lambda t}$, and an analytic continuation.

(iii) Prove that the martingale $e^{B_t - \frac{1}{2}t}$ stopped at σ_a is uniformly integrable.

(iv) For $a > 0$ and $b > 0$, define $\sigma_{a,b} = \inf\{t \geq 0 : B_t < bt - a\}$. Prove that $\sigma_{a,b} \overset{\text{law}}{=} b^{-2}\sigma_{ab,1}$ and

$$\mathbb{E}(e^{\frac{1}{2}b^2\sigma_{a,b}}) = e^{ab}.$$

(v) For $b < 1$, prove that $\mathbb{E}(e^{\frac{1}{2}\sigma_{1,b}}) = \infty$.

Exercise 2. Let B be a Brownian motion. For $n \geq 0$, let

$$H_n(x, y) = (\partial_\alpha)^n \mid_{\alpha=0} e^{\alpha x - \frac{\alpha^2}{2} y}.$$

Prove that for $n = 1, 2, 3$, $H_n(B_t, t)$ is a martingale. Prove this for any n.

Exercise 3. Let $X_t = \int_0^t (\sin s)\mathrm{d}B_s$. Prove that this is a Gaussian process. What are $\mathbb{E}(X_t)$ and $\mathbb{E}(X_s X_t)$? Prove that

$$X_t = (\sin t)B_t - \int_0^t (\cos s)B_s \mathrm{d}s.$$

Exercise 4. Prove that if f is a deterministic continuous square integrable function,

$$\mathbb{E}\left(B_t \int_0^\infty f(s)\mathrm{d}B_s\right) = \int_0^t f(s)\mathrm{d}s.$$

Exercise 5. If M is a continuous local martingale with $M_0 = 0$, prove that almost surely

$$\{(e^{M_t - \frac{1}{2}\langle M\rangle_t})_{t=\infty} = 0\} = \{\langle M\rangle_\infty = \infty\},$$

i.e. the symmetric difference between both events has measure 0.

Exercise 6. Let B^1 and B^2 be independent Brownian motions, defined on the same probability space. Let

$$X_t = e^{B_t^1} \int_0^t e^{-B_s^1} \mathrm{d}B_s^2,\ Z_t = \sinh B_t^1.$$

Prove that both processes have the same law.