Probability, homework 8, due May 6th.

Exercise 1. Prove that the following processes are martingales with respect to the Brownian filtration.

(i) $X_t = e^{\frac{t}{2}} \cos(B_t)$;
(ii) $X_t = (B_t + t)e^{-B_t - \frac{t}{2}}$.

Exercise 2. Let X be a stochastic proces starting at X_0 and satisfying $\mathrm{d}X_t = (-\alpha X_t + \beta)\mathrm{d}t + \sigma \mathrm{d}B_t$, where $\alpha > 0$.

(i) Give an explicit expression for X_t.
(ii) Calculate $\mathrm{Cov}(X_s, X_t)$ for any $s < t$.

Exercise 3. Assume the process $(X_t)_{t \geq 0}$ satisfies $\mathrm{d}X_t = X_t(\mu_t \mathrm{d}t + \sigma_t \mathrm{d}B_t)$ for some Brownian motion B which corresponds to the Wiener measure $\mathbb{P}$.

(i) Prove that $X_t e^{-\int_0^t \mu_s \mathrm{d}s}$ is a local martingale under $\mathbb{P}$.
(ii) Find a probability $\mathbb{Q}$ under which X is a local martingale.
(iii) Find a probability $\widetilde{\mathbb{Q}}$ under which X^{-1} is a local martingale.

Exercise 4. Consider the general equation

$$\mathrm{d}X_t = (c(t) + d(t)X_t)\mathrm{d}t + (e(t) + f(t)X_t)\mathrm{d}B_t,\ X_0 = 0.$$

where c, d, e, f are deterministic. We try to find a solution of type $X = X^{(1)}X^{(2)}$ where

$$\mathrm{d}X_t^{(1)} = d(t)X_t^{(1)}\mathrm{d}t + f(t)X_t^{(1)}\mathrm{d}B_t,\ X_0^{(1)} = 1,$$
$$\mathrm{d}X_t^{(2)} = a(t)\mathrm{d}t + b(t)\mathrm{d}B_t,\ X_0^{(2)} = X_0,$$

and a, b are stochastic processes to be chosen.

(i) Prove that $X_t^{(1)} = e^{\int_0^t f(s)\mathrm{d}B_s - \frac{1}{2}\int_0^t f(s)^2 \mathrm{d}s + \int_0^t d(s)\mathrm{d}s}$ is a solution.
(ii) Identify necessary formulas for a and b.
(iii) Conclude a general formula for the solution of the initial equation.

Exercise 5. For a given Brownian motion B, let X be a solution of

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}B_t + b(X_t)\mathrm{d}t,\ X_0 = x,$$

and $X^{(n)}$ be a solution of

$$\mathrm{d}X_t = \sigma^{(n)}(X_t)\mathrm{d}B_t + b^{(n)}(X_t)\mathrm{d}t,\ X_0 = x,$$

where all functions are Lipschitz with the same absolute constant independent of n. Assume pointwise convergence of $\sigma^{(n)}$ to σ, and of $b^{(n)}$ to b. Prove that for any $t > 0$, as $n \to \infty$,

$$\mathbb{E}\left(\sup_{[0,t]} |X_s - X_s^{(n)}|^2\right) \to 0.$$

Exercise 6. *(bonus)* Let B be a Brownian motion, $a > 0$, $\gamma \in \mathbb{R}$, and $S_{a,\gamma} = \inf\{t \geq 0 \mid |B_t + \gamma t| = a\}$. Are $S_{a,\gamma}$ and $B_{S_{a,\gamma}} + \gamma S_{a,\gamma}$ independent?