Probability, training for the final exam

Exercise 1. Let $X_n, n \geq 0$, be independent random variables. Assume that $\mathbb{E}(X_j) = 0$ and there exists a $\beta > 0$ such that $\mathbb{E}(|X_j|^2) = j^{-\beta}$ for any $j \geq 1$. Let $S_n = \sum_{k=1}^{n} X_k$. For which values of β does S converge almost surely? Prove it.

Exercise 2. Let B be a Brownian motion starting at $x > 0$, and $T_0 = \inf\{s \geq 0 : B_s = 0\}$. What is the distribution of $\sup_{t \leq T_0} B_t$? You can apply a stopping time theorem to a positive martingale.

Exercise 3. Let B be a standard Brownian motion and $M_t = \max_{0 \leq s \leq t} B_s$.

(1) Explain why M_t has the same distribution as $\sqrt{t} M_1$.
(2) What is the density of M_t?

Exercise 4. Let $(S_n)_{n \geq 0}$ be a standard random walk.

(i) State Donsker's theorem for $(S_n)_{n \geq 0}$.
(ii) As $N \to \infty$, find the asymptotics for

$$\mathbb{E}\left(\max_{N/2 < n < N} |S_n|\right).$$

Exercise 5. Sketch the proof that the Brownian motion is transcient in dimension $d \geq 3$.

Exercise 6. Let $a > 0$, $\gamma \geq 0$, and $T_{a,\gamma} = \inf\{t \geq 0 \mid B_t + \gamma t = a\}$. Prove that the density of $T_{a,\gamma}$ with respect to the Lebesgue measure on $\mathbb{R}_+$ is

$$\frac{a}{\sqrt{2\pi t^3}} e^{\frac{-(a-\gamma t)^2}{2t}}.$$

If you solve the case $\gamma = 0$, you get 3/4 of the points.

Exercise 7. Let B be a standard Brownian motion. Justify the following stochastic differential equation has only one solution. In which sense? What does that mean?

$$\mathrm{d}X_t = X_t \mathrm{d}t + (1 - e^{-|X_t|}) X_t \mathrm{d}B_t.$$

Exercise 8.

(i) State the Feynman-Kac theorem.
(ii) Let B be a Brownian motion starting at 0. What partial differential equation does $\mathbb{E}\left(e^{\int_t^T B_s^2 \mathrm{d}s}\right)$ satisfy? What are the boundary conditions?

Exercise 9. What stochastic differential equation does $(e^{-t} B_{e^{2t}})_{t \geq 0}$ satisfy? What is the name of this process?

Exercise 10. Let a be a given deterministic function. Calculate

$$\mathbb{E}\left(e^{\int_0^t a(s) B_s^2 \mathrm{d}s}\right).$$

Exercise 11. Let X and Y be independent Brownian motions.

1) Assume $X_0 = Y_0 = 0$, and note $T_a = \inf\{t \geq 0 \mid X_t = a\}$ for $a > 0$. Prove that T_a has the same law as $a^2/\mathcal{N}^2$, where $\mathcal{N}$ is a standard normal variable.

2) Prove that Y_{T_a} has the same law as $a\,C$, where the Cauchy random variable C is defined through its density with respect to the Lebesgue measure,

$$\frac{1}{\pi(1 + x^2)}.$$

3) Let $(X_0, Y_0) = (\epsilon, 0)$, where $0 < \epsilon < 1$. Note $Z_t = X_t + \mathrm{i}Y_t$. Justify that the winding number

$$\theta_t = \frac{1}{2\pi} \arg Z_t$$

can be properly defined, continuously from $\theta_0 = 0$. Let $T^{(\epsilon)} = \inf\{t \geq 0 \mid |Z_t| = 1\}$. Prove that

$$\frac{\theta_{T^{(\epsilon)}}}{\log \epsilon}$$

is distributed as $\frac{1}{2\pi} C$, C being a Cauchy random variable.

4) Let $(X_0, Y_0) \neq (0,0)$ and define as previously $Z_t = X_t + \mathrm{i} Y_t$ and $\arg Z_t$ continuously from $\arg Z_0 \in [0, 2\pi)$. Prove that, as $t \to \infty$,

$$\frac{2 \arg Z_t}{\log t} \xrightarrow{\text{law}} C.$$