

Probability, homework 1, solutions.

Exercise 1. Let $(\mathcal{G}_\alpha)_{\alpha \in A}$ be an arbitrary family of σ -fields defined on an abstract space Ω , with A possibly uncountable. Show that $\cap_{\alpha \in A} \mathcal{G}_\alpha$ is also a σ -field.

Solution. We need to check that

- (i) $\emptyset \in \cap_{\alpha \in A} \mathcal{G}_\alpha$,
- (ii) if $A \in \cap_{\alpha \in A} \mathcal{G}_\alpha$, then $A^c \in \cap_{\alpha \in A} \mathcal{G}_\alpha$,
- (iii) if $(A_i)_{i \geq 1}$ is a sequence in $\cap_{\alpha \in A} \mathcal{G}_\alpha$, then $\cup_{i \geq 1} A_i \in \cap_{\alpha \in A} \mathcal{G}_\alpha$ and $\cap_{i \geq 1} A_i \in \cap_{\alpha \in A} \mathcal{G}_\alpha$.

Let us check just the last statement on intersections, all others are proved similarly and left to the reader. If $A_i \in \cap_{\alpha \in A} \mathcal{G}_\alpha$, for any α we have $A_i \in \mathcal{G}_\alpha$. As $\mathcal{G}_\alpha$ is a σ -algebra this implies $\cap_{i \geq 1} A_i \in \mathcal{G}_\alpha$. As this is true for any α , we have $\cap_{i \geq 1} A_i \in \cap_{\alpha \in A} \mathcal{G}_\alpha$.

Exercise 2. Let $\emptyset \subsetneq A \subsetneq B \subsetneq \Omega$ (these are strict inclusions). What is the σ -field generated by $\{A, B\}$?

Solution. It is clear that $\{\emptyset, A, B, A^c, B^c, B - A, (B - A)^c, \Omega\} \subset \sigma(\{A, B\})$. On the other hand, stability of this left hand side by complement, union, intersection can be checked by hand.

Exercise 3. Let $\mathcal{F}, \mathcal{G}$ be σ -fields for the same Ω . Is $\mathcal{F} \cup \mathcal{G}$ a σ -field?

Solution. Consider $\mathcal{F} = \{\emptyset, A, A^c, \Omega\}$, $\mathcal{G} = \{\emptyset, B, B^c, \Omega\}$ with A, B nonempty sets such that $A \cup B \neq \Omega$. Then $\mathcal{F} \cup \mathcal{G}$ contains A, B but not $A \cup B$ so it is not a σ -field.

Exercise 4. For $\Omega = \mathbb{N}$ and $n \geq 0$, let $\mathcal{F}_n = \sigma(\{\{0\}, \dots, \{n\}\})$. Show that $(\mathcal{F}_n)_{n \geq 0}$ is a non-decreasing sequence but that $\cup_{n \geq 0} \mathcal{F}_n$ is not a σ -field.

Solution. As $\{\{0\}, \dots, \{n\}\} \subset \{\{0\}, \dots, \{n+1\}\}$, by definition of σ we have $\mathcal{F}_n \subset \mathcal{F}_{n+1}$, i.e. the sequence is non-decreasing.

Assume $\mathcal{A} := \cup_{n \geq 0} \mathcal{F}_n$ is a σ -field. Then the set $A = \{1, 2, \dots\}$ of all positive integers is in $\mathcal{A}$ because $A = \cup_{n \geq 1} \llbracket 1, n \rrbracket$ and $\llbracket 1, n \rrbracket \in \mathcal{F}_n \subset \mathcal{A}$. Hence there exists n_0 such that $A \in \mathcal{F}_{n_0}$. But this contradicts $n_0 + 1 \in A$ and $n_0 + 1$ not contained in any element of $\mathcal{F}_{n_0}$.

Exercise 5. Let Ω be an infinite set (countable or not). Let $\mathcal{A}$ be the set of subsets of Ω that are either finite or with finite complement in Ω . Prove that $\mathcal{A}$ is a field but not a σ -field.

Solution. The set $\mathcal{A}$ is a field: clearly the null and full sets are in it, $\mathcal{A}$ is stable by immediate definition, and if A and B are in $\mathcal{A}$, then $A \cup B$ is either finite (when A

and B are finite) or its complement $A^c \cap B^c$ is finite (when A or B is infinite with finite complement), so that $A \cup B \in \mathcal{A}$.

However, $\mathcal{A}$ is not a σ -field. Indeed, let $\omega_1, \omega_2, \dots$ be an infinite sequence of disjoint elements in Ω . Then $\{\omega_2, \omega_4, \omega_6, \dots\}$ is the countable union of elements in $\mathcal{A}$ (the $\{\omega_{2i}\}$'s), but it is neither finite nor with finite complement, thus it cannot be in $\mathcal{A}$.

Exercise 6. Can you build an infinite, countable σ -field?

Solution. Consider $(\Omega, \mathcal{A})$ with $\mathcal{A}$ a σ -field. For any $x \in \Omega$, define

$$\dot{x} = \bigcap_{A \in \mathcal{A}: x \in A} A.$$

Then $\{\dot{x}, x \in \Omega\}$ (eliminating repetitions) form a partition of Ω (uses stability of the field by complement).

If $\mathcal{A}$ is countable, $\dot{x}$ is an element of $\mathcal{A}$, as a countable intersection of elements in $\mathcal{A}$. Thus $\mathcal{A}$ contains all sets of type $\dot{x}$, and all countable unions of such sets. In fact, $\mathcal{A}$ consists exactly in such unions (easy to prove).

If the number of atoms is finite, then $\mathcal{A}$ is finite. If the number of atoms is infinite, then the number of their countable unions is uncountable (cf a map to Cantor diagonalization), a contradiction.

We have proved that $\mathcal{A}$ is either finite or uncountable.

Exercise 7. A monotone class is a collection $\mathcal{M}$ of sets closed under both monotone increasing and monotone decreasing (i.e. if $A_i \in \mathcal{M}$ and either $A_i \uparrow A$ or $A_i \downarrow A$ then $A \in \mathcal{M}$)

Prove that if $\mathcal{A} \subset \mathcal{M}$ with $\mathcal{A}$ a field and $\mathcal{M}$ a monotone class, then $\sigma(\mathcal{A}) \subset \mathcal{M}$.

Solution. Clearly, any field which is a monotone class must be a σ -field. Moreover, we will prove below the following:

(Claim) the intersection $m(\mathcal{A})$ of all monotone classes containing a field $\mathcal{A}$ is both a field and a monotone class.

Hence $m(\mathcal{A})$ is a σ -algebra. Since $\mathcal{A} \subset m(\mathcal{A})$ this implies $\sigma(\mathcal{A}) \subset m(\mathcal{A})$, which concludes the proof by noting $m(\mathcal{A}) \subset \mathcal{M}$.

We now prove the above (Claim). The fact that $m(\mathcal{A})$ is a monotone class is straightforward from the definition. We now show that $m(\mathcal{A})$ is a field. Inclusion of null and full sets is trivial.

For the stability by complement: Consider $\mathcal{B} = \{A : A^c \in m(\mathcal{A})\}$. As $m(\mathcal{A})$ is monotone, so is $\mathcal{B}$. Hence $\mathcal{B}$ is a monotone class containing $\mathcal{A}$, hence $m(\mathcal{A}) \subset \mathcal{B}$, which implies stability by complement.

For stability by union, let

$$\mathcal{G}_1 = \{A : A \cup B \in m(\mathcal{A}) \text{ for all } B \in \mathcal{A}\}.$$

Then $\mathcal{G}_1$ is a monotone class (simply checked by careful writing based on definitions) containing $\mathcal{A}$ hence $m(\mathcal{A}) \subset \mathcal{G}_1$. Now let

$$\mathcal{G}_2 = \{B : B \cup A \in m(\mathcal{A}) \text{ for all } A \in m(\mathcal{A})\}.$$

Then $\mathcal{G}_2$ is a monotone class (again just by a careful direct writing). We have $\mathcal{A} \subset \mathcal{G}_2$ because for $B \in \mathcal{A}$ and $A \in m(\mathcal{A}) \cup \mathcal{G}_1$ we have $A \cup B \in m(\mathcal{A})$ by definition of $\mathcal{G}_1$. From $\mathcal{A} \subset \mathcal{G}_2$ we deduce that $m(\mathcal{A}) \subset \mathcal{G}_2$, so for any $A, B \in m(\mathcal{A})$ we have $A \cup B \in m(\mathcal{A})$ by definition of $\mathcal{G}_2$.

Exercise 8. Let $\mathbb{P}$ be a probability measure on Ω , endowed with a σ -field $\mathcal{A}$.

- (i) What is the meaning of the following events, where all A_n 's are elements of $\mathcal{A}$?

$$\liminf_{n \rightarrow \infty} A_n = \bigcup_{n \geq 1} \bigcap_{k \geq n} A_k, \quad \limsup_{n \rightarrow \infty} A_n = \bigcap_{n \geq 1} \bigcup_{k \geq n} A_k.$$

- (ii) Prove that $\limsup_{n \rightarrow \infty} A_n$ and $\liminf_{n \rightarrow \infty} A_n$ are in $\mathcal{A}$.
 (iii) In the special case $\Omega = \mathbb{R}$, for any $p \geq 1$, let

$$A_{2p} = \left[-1, 2 + \frac{1}{2p}\right), \quad A_{2p+1} = \left(-2 - \frac{1}{2p+1}, 1\right].$$

What are $\liminf_{n \rightarrow \infty} A_n$ and $\limsup_{n \rightarrow \infty} A_n$?

- (iv) Prove that the following always holds:

$$\mathbb{P}\left(\liminf_{n \rightarrow \infty} A_n\right) \leq \liminf_{n \rightarrow \infty} \mathbb{P}(A_n), \quad \mathbb{P}\left(\limsup_{n \rightarrow \infty} A_n\right) \geq \limsup_{n \rightarrow \infty} \mathbb{P}(A_n).$$

Solution. (i) The event $\liminf_{n \rightarrow \infty} A_n$ means that A_n always occurs for large enough n . The event $\limsup_{n \rightarrow \infty} A_n$ means that infinitely many A_n 's happen.

(ii) Elementary by countable intersections and unions.

(iii) For any n , $\bigcap_{k \geq n} A_k = [-1, 1]$, so $\liminf_{n \rightarrow \infty} A_n = [-1, 1]$. For any n , $\bigcup_{k \geq n} A_k = (-2 - a_n, 2 + b_n)$ with a_n, b_n positive converging to 0 as $n \rightarrow \infty$, so $\limsup_{n \rightarrow \infty} A_n = [-2, 2]$.

(iv) For any $k \geq n$ we have $\mathbb{P}(A_n) \geq \mathbb{P}(\cap_{k \geq n} A_k)$ so $\inf_{n \geq m} \mathbb{P}(A_n) \geq \inf_{n \geq m} \mathbb{P}(\cap_{k \geq n} A_k) = \mathbb{P}(\cap_{k \geq m} A_k)$. Take the limit $m \rightarrow \infty$: The LHS converges to $\liminf_{n \rightarrow \infty} \mathbb{P}(A_n)$ by definition of the $\liminf$, and the RHS converges to $\mathbb{P}(\liminf_{n \rightarrow \infty} A_n)$ as the events $\cap_{k \geq m} A_k$ increase to $\liminf_{n \rightarrow \infty} A_n$.

The proof for $\limsup$ is similar.

Exercise 9. The symmetric difference of two events A and B , denoted $A \triangle B$, is the event that precisely one of them occurs: $A \triangle B = (A \cup B) \setminus (A \cap B)$.

- (i) Write a formula for $A \triangle B$ that only involves the operations of union, intersection and complement, but no set difference.
 (ii) Define $d(A, B) = \mathbb{P}(A \triangle B)$. Show that for any three events A, B, C ,

$$d(A, B) + d(B, C) - d(A, C) = 2(\mathbb{P}(A \cap B^c \cap C) + \mathbb{P}(A^c \cap B \cap C^c)).$$

- (iii) Assume $A \subset B \subset C$. Prove that $d(A, C) = d(A, B) + d(B, C)$.

Solution. (i) $A \triangle B = (A \cap B^c) \cup (B \cap A^c)$

(ii) Note that $A \cap B^c$ and $B \cap A^c$ are disjoint so $\mathbb{P}(A \triangle B) = \mathbb{P}(A \cap B^c) + \mathbb{P}(B \cap A^c) = \mathbb{P}(A \cap B^c \cap C) + \mathbb{P}(B \cap A^c \cap C^c) + \mathbb{P}(A \cap B^c \cap C) + \mathbb{P}(B \cap A^c \cap C^c)$. Injecting this formula and the analogues for $\mathbb{P}(A \triangle C), \mathbb{P}(B \triangle C)$ gives the answer.

(iii) Immediate from the previous question as the sets on the RHS are empty.

Exercise 10. Prove the Bonferroni inequalities: if $A_i \in \mathcal{A}$ is a sequence of events, then

$$(i) \quad \mathbb{P}(\cup_{i=1}^n A_i) \geq \sum_{i=1}^n \mathbb{P}(A_i) - \sum_{i < j} \mathbb{P}(A_i \cap A_j),$$

$$(ii) \quad \mathbb{P}(\cup_{i=1}^n A_i) \leq \sum_{i=1}^n \mathbb{P}(A_i) - \sum_{i < j} \mathbb{P}(A_i \cap A_j) + \sum_{i < j < k} \mathbb{P}(A_i \cap A_j \cap A_k).$$

Solution. For any $x \in \Omega$ we claim that

$$\mathbb{1}_{x \in \cup_{i=1}^n A_i} \geq \sum_i \mathbb{1}_{x \in A_i} - \sum_{i < j} \mathbb{1}_{x \in A_i \cap A_j}. \quad (0.1)$$

Indeed, if x is in exactly m sets A_i 's, for $m = 0$ the above relation is $0 \geq 0$ and if $m \geq 1$ this is

$$1 \geq m - \binom{m}{2},$$

which is true. Integrating (0.1) w.r.t $\mathbb{P}$ gives the result.

The same type of reasoning holds for (ii), stopping the binomial expansion at third order.