

Probability, homework 2, due September 16.

Exercise 1. Let $\mathcal{A}$ be a σ -algebra, $\mathbb{P}$ a probability measure and $(A_n)_{n \geq 1}$ a sequence of events in $\mathcal{A}$ which converges to A . Prove that

- (i) $A \in \mathcal{A}$;
- (ii) $\lim_{n \rightarrow \infty} \mathbb{P}(A_n) = \mathbb{P}(A)$.

Exercise 2. Suppose a distribution function F is given by

$$F(x) = \frac{1}{4} \mathbb{1}_{[0, \infty)}(x) + \frac{1}{2} \mathbb{1}_{[1, \infty)}(x) + \frac{1}{4} \mathbb{1}_{[2, \infty)}(x)$$

What is the probability of the following events, $(-1/2, 1/2)$, $(-1/2, 3/2)$, $(2/3, 5/2)$, $(3, \infty)$?

Exercise 3. Let μ be the Lebesgue measure on $\mathbb{R}$. Build a sequence of functions $(f_n)_{n \geq 0}$, $0 \leq f_n \leq 1$, such that $\int f_n d\mu \rightarrow 0$ but for any $x \in \mathbb{R}$, $(f_n(x))_{n \geq 0}$ does not converge.

Exercise 4. Let X be a random variable in $L^1(\Omega, \mathcal{A}, \mathbb{P})$. Let $(A_n)_{n \geq 0}$ be a sequence of events in $\mathcal{A}$ such that $\mathbb{P}(A_n) \xrightarrow{n \rightarrow \infty} 0$. Prove that $\mathbb{E}(X \mathbb{1}_{A_n}) \xrightarrow{n \rightarrow \infty} 0$.

Exercise 5. Let $(d_n)_{n \geq 0}$ be a sequence in $(0, 1)$, and $K_0 = [0, 1]$. We define iteratively $(K_n)_{n \geq 0}$ in the following way. From K_n , which is the union of closed disjoint intervals, we define K_{n+1} by removing from each interval of K_n an open interval, centered at the middle of the previous one, with length d_n times the length of the previous one. Let $K = \bigcap_{n \geq 0} K_n$ (K is called a Cantor set).

(a) Prove that K is an uncountable compact set, with empty interior, and whose points are all accumulation points

(b) What is the Lebesgue measure of K ?

Exercise 6. Let X be a nonnegative random variable. Prove that $\mathbb{E}(X) < +\infty$ if and only if $\sum_{n \in \mathbb{N}} \mathbb{P}(X \geq n) < \infty$.

Exercise 7. *Convergence in measure.* Let $(\Omega, \mathcal{A}, \mu)$ be a probability space. and $(f_n)_{n \geq 1}$, $f : \Omega \rightarrow \mathbb{R}$ measurable (for the Borel σ -field on $\mathbb{R}$). We say that $(f_n)_{n \geq 1}$ converges in measure to f if for any $\varepsilon > 0$ we have

$$\mu(|f_n - f| > \varepsilon) \xrightarrow{n \rightarrow \infty} 0.$$

- (i) Show that $\int |f - f_n| d\mu \rightarrow 0$ implies that f_n converges to f in measure. Is the reciprocal true?
- (ii) Show that if $f_n \rightarrow f$ μ -almost surely, then $f_n \rightarrow f$ in measure. Is the reciprocal true?
- (iii) Show that if $f_n \rightarrow f$ in measure, there exists a subsequence of $(f_n)_{n \geq 1}$ which converges μ -almost surely.
- (iv) (*A stronger dominated convergence theorem*) We assume that $f_n \rightarrow f$ in measure and $|f_n| \leq g$ for some integrable $g : \Omega \rightarrow \mathbb{R}$, for any $n \geq 1$.
 - (a) Show that $|f| \leq g$ μ -a.s.
 - (b) Deduce that $\int |f_n - f| d\mu \rightarrow 0$.

Exercise 8. Consider a probability space $(\Omega, \mathcal{A}, \mu)$ and $(A_n)_n$ a sequence in $\mathcal{A}$. Let $f : \Omega \rightarrow \mathbb{R}$ be measurable (for the Borel σ -field on $\mathbb{R}$) such that $\int_{\Omega} |\mathbb{1}_{A_n} - f| d\mu \rightarrow 0$ as $n \rightarrow \infty$. Prove that there exists $A \in \mathcal{A}$ such that $f = \mathbb{1}_A$ μ -a.s., i.e. $\mu(f = \mathbb{1}_A) = 1$.

Exercise 9. Consider a probability space $(E, \mathcal{A}, \mu)$ and $f_n : E \rightarrow \mathbb{R}$ measurable, $n \geq 1$. Assume $f_n \rightarrow f$ μ -almost surely. Prove that for any $\varepsilon > 0$ there exists a set $A \in \mathcal{A}$ such that $\mu(A) < \varepsilon$ and the convergence $f_n \rightarrow f$ is uniform on A^c .